The Minimalist Foundation

and its impact on the working mathematician

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Aim today

My aim today is to show that the study of foundations is convenient also for purely technical purposes.

Computing with Infinite Data was Brouwer's main motivation.
data = construction by somebody else

paradigm in mathematics = a conception of the meaning and foundation of mathematics
B. Pourciau, Intuitionism as a (failed) Kuhnian revolution in mathematics, Stud. Hist.Phil. Sci. 2001

adopting a new paradigm brings new understanding, new results and perhaps solution to old problems.

In particular, the absence of axiom of unique choice (and hence the distinction between operation and function) allows one to conceive choice sequence (or streams) as ideal points of a pointfree Baire space.

The problem of foundations

- **What is mathematics?** Question put seriously in:
  - ancient Greece
  - 19th century, Europe Europe, is this relevant?
- Problem of foundations: what is the **meaning** of mathematics?

  intuition had been challenged by:
  - non-euclidean geometry: loss of absolute truth in geometry
  - abstract algebra (to cope with complexity)
  - rigorization of analysis ("pathological curves",...)
- Cantor, Dedekind, Frege, Peano: naive set theory
- Paradoxes, i.e. **contradictions**: Burali-Forti 1896, Russel 1901, ...
- Crisis of foundations
- Traditional ways out:
  - logicism (Frege, Russell, Whitehead, Principia Mathematica 1911)
  - constructivism (Kronecker, Borel, Poincaré, Brouwer, Heyting,...)
  - formalism (Hilbert, Zermelo,...)

Hilbert program, Enriques' criterion and Gödel's theorems

Hilbert's program: consistency of ZFC, a finitary proof

for Zermelo, Fraenkel plus axiom of Choice

Brouwer : consistency is a not sufficient to give meaning

Enriques' criterion
If then you would not lose yourself in a dream devoid of sense, you should not forget the supreme condition of positivity, by means of which the critical judgement must **affirm or deny**, *in the last analysis,* **facts**

F. Enriques, *Problemi della scienza*, 1906, English transl. 1914

But: ZFC does not satisfy Enriques' criterion.

We don't have a proof of formal consistency of ZFC,

and most probably we will never have one:

$ZFC \nvdash Con(ZFC)$ by Gödel's 2nd incompleteness theorem

Common paradigm today

Somehow paradoxically... the common paradigm is:

Bourbaki's attitude = denial of the problem

platonist on weekdays, formalist on sundays   split mind

... when philosophers attack... we rush to hide behind formalism and say "mathematics is just a combination of meaningless symbols"... we are left in peace... with the feeling each mathematician has that he is working with something real. This sensation is probably an illusion, but is very convenient. That is Bourbaki's attitude toward foundations.

J. A. Dieudonné, 1970, see Davis-Hersh *The mathematical experience*, 1981

Formally   classical logic and axiomatic set theory ZFC, ignoring Gödel.

Many mathematicians say they follow ZFC without being aware of its **problems**.
One assumes existence (where?) of objects satisfying ZFC. So there is a meaning, but we do not know which.
An **act of faith** remains necessary

ZFC was meant to be the solution, it has become part of the problem...

Synthesis

Thesis: classical approach via ZFC

Antithesis: only mathematics with a computational meaning
(Bishop, Martin-Löf)

Synthesis: After over 100 years, it is the right time to look for a synthesis

Bishop's book *Foundations of constructive analysis (FCA)*, 1967, showed:
constructive mathematics does **not** depend on Brouwer's subjective views

after FCA constructive mathematics has become a rich and lively research field

Not successful among mathematicians (safely less than 2%) because:

1. fear that much of mathematics is cut off
2. motivations are not clear, still partly subjective

50 years after Bishop, we wish to make constructivism **stronger**:

more **solid**, more **general**, more **appealing**

Where should we look for help?

A change of paradigm - on the shoulder of giants

Epochal changes after 1967 provide motivations and support.
Outside mathematics:

• **evolution** is now commonly accepted in science, **except** mathematics.
Main challenge: pass from a static, transcendent view of mathematics (see page 1 of FCA) to a **dynamic**, **evolutionary**, human one.

this is the change of paradigm

• The **power of computers** has enormously increased. The role of computers (**proof assistants**) in mathematical research will increase.
It requires **fully detailed formal systems** for foundations.

• New information technology means an **intensely connected world**.
Old views (absolute truths) create extremely high tensions.
We need **pluralism** of views, basing their strength on
internal awareness rather than external authority or force.

Tai Ji Quan rather than Boxing

Inside mathematics:

• new branches have been created

• other branches (algebra, topology,...) have been constructivized

We feel more relaxed.

Comparison with the the hottest trend today

Homotopy type theory Hott, alias Univalent Foundation

- accepts mathematics as "given", as in the classical paradigm. Only this attitude can explain why it puts as an advanced discipline, such as homotopy theory, at the base

- keeps silent about pluralism

- It is not satisfactory also for its original aim, i.e. certification of mathematics, since it does not have two levels of abstraction (intensional/extensional)

Dynamic constructivism

let's go back to the question: what is mathematics?, and look at facts:

- it is **simpler** and **more effective** to manipulate symbols than things:

$$\begin{array}{ccccc} & & \text{mathematical manipulation} & & \\ & & \longrightarrow & & \\ \text{abstraction} & \Big\uparrow & & \Big\downarrow & \text{application} \\ & & \longrightarrow & & \\ & & \text{reality} & & \end{array}$$

- every culture has its own mathematics (it is useful to man for survival, it is a continuation of natural **evolution**).

some consequences:

- nothing is given, every notion is the result of an abstraction
- many ways to abstract = many kinds of mathematics, pluralism

  foundational system = choice of what kind of information is relevant
- application is part of mathematics
- the question "what mathematical entities are" replaced by: why and how we construct them, how we communicate them, to what we can apply them, etc.
- objectivity is a result, not a cause; dynamic, evolutionary view of mathematics
- all other sciences are based on evolution; only a wish that math is different
- mathematics is the exploration of notions and structures of our abstract (reliable) thought which can be useful to understand the world.

Properties of a satisfactory foundation

- **Trustable**

  We trust in its consistency by a **proof**, not by faith or feelings

- **Applicable, meaningful**

  Enriques' criterion: application = facts of which we speak

  In rigorous terms: **realizability interpretation**

  It must allow **formalization of mathematics** in a proof assistant

- **Precise, universal**

  every notion has a clear meaning

  all meaningful conceptual distinctions are preserved

  minimal in assumptions **hence maximal** in distinctions

  a **framework for pluralism**: all foundations can be expressed

  final setting for reverse mathematics

All of this is possible!
the **M**inimalist **F**oundation **MF** j.w.w. Milly Maietti,
agrees with the perspective of dynamic constructivism

Adopting dynamic constructivism in practice means doing mathematics in MF, or equivalently adhering to the following four principles.

1. Cultivate pluralism in mathematics and foundations.
Different styles in abstraction, which means different foundations, produce different kinds of mathematics and should be respected.
constructivism is not constructivization of classical mathematics;
new definitions, corresponding to a different way of abstracting.

MF is compatible with the most relevant foundations: each of them is obtained as an extension of MF.

2. Accept open notions and incomplete theories. The construction of mathematics is a never-ending process and nothing is given in advance
whatever assumes a blocked process is rejected,
no fixed universes of all sets, or of all subsets, or of all propositions.

Many notions are open-ended, intrinsically incomplete
a source of a more relaxed view and a deeper understanding.
consistency of MF becomes a theorem, contrary to ZFC.

3. Preserve all conceptual distinctions (no reductionism).

the achievements of mathematics (not only theorems or solutions to problems but also definitions, intuitions, conceptual distinctions, etc.) are the result of human struggle and thus become precious and must be kept, without reducing all to a single notion, like that of set.

As a consequence, many more primitive notions than usual

In particular: set, collection and proposition,
also in their form under assumptions
(which produce the notions of operation, subset, relation, function, etc.).

4. Preserve all different levels of abstraction.

different levels of abstraction, such as the computational, set-theoretic and algebraic modes

distinction between language and metalanguage

In particular, intensional aspects live together with extensional ones: MF has two levels of abstraction. see Maietti's talk

Minimalism

minimalist in assumptions = maximalist in conceptual distinctions

- LEM (*Law of Excluded Middle*) $\varphi \vee \neg\varphi$ true for all propositions φ

The matter is not whether LEM is true or not, but whether we care to distinguish positive and negative (classical) notion of existence.
Assuming LEM one derives $\exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$. So if one is unwilling or unable to give up LEM, it means one does not care about the distinction $\exists \neq \neg\forall\neg$, or is unable to see it.

- PSA (*Power Set Axiom*) if X is a set, then also $\mathcal{P}X$ is a set

Rejecting PSA allows to preserve a constructive conception of sets.

sets: real, effective (finite number of rules to generate all elements = inductively generated); **stable in time, fully communicable**

collections: ideal (no induction); e.g. *Prop*, $\mathcal{P}X$, $\mathbb{R}$

open ended, can change tomorrow

Validity of PSA means that one cannot separate constructible sets from other.
NB it's not a matter of words

In particular, we obtain a constructive version of $\mathcal{P}(X)$ as the **collection of subsets** of X, with extensional equality.

Where: subset of $X =_{def}$ proposition with one argument in X

Minimalism

- $\texttt{AC!}$ (Axiom of unique Choice)
  $(\forall x \in X)(\exists! y \in Y)R(x,y) \to (\exists f : X \to Y)(\forall x \in X)R(x, fx)$

Rejecting $\texttt{AC!}$ (and hence Axiom of Choice $\texttt{AC}$) allows to keep the distinction between:

function $\quad (\forall x \in X)(\exists! y \in Y)R(x,y)$ total and singlevalued relation

we know that the value is unique, but don't know which

the common notion in set theory

operation $\quad p(x) \in Y \ (x \in X)$ dependent family of elements

we know how to produce the value on every input

a common notion in constructive mathematics (Bishop, type theory)

Keeping the distinction function/operation

If we wish to keep the distinction between function and operation, we must keep validity of `AC!` under control.

How can we make `AC!` not valid in MF?

We need to distinguish

weak existence $\exists$

$\exists x \varphi(x)$ true when we have a guarantee that a witness c can eventually be found, also when no operation providing it is available.

strong existence Σ + prop-as-sets

Examples of weak existence:

the holy man to the pilgrim: "if you will be tomorrow in the same place and at the same time as today, your wife is saved"

two swipe cards, only one is active, I don't know which

To keep this distinction, we need propositions $\neq$ sets.

function $=$ operation given by somebody else, without giving instructions

Novelties in mathematics

"the book" shows in practice that a lot of mathematics (all of topology) can be done in this minimalist way

Note: to generate pointfree topologies, we need to assume the principle:

ICAS: generation by induction and coinduction from an axiom-set

price: one has to start again from the beginning: main task is to find correct constructive definitions

Nothing good of ZFC is a priori out of reach.

most interesting, fascinating **reward**: several novelties emerge which were **hidden** by stronger foundations, using `PSA`, `LEM`

surprise: the **extra information** which we must keep has a clear logical structure, it is not "code"

in practice: we start by keeping the base of a topological space...

... and see that this **improves on the structure** (of notions, of results, of our understanding,...)

duality and symmetry in topology 1

To produce a topology ΩX on a set X without $\mathtt{PSA}$ it is necessary to start from a base for open subsets $\text{ext}\,(a) \subseteq X$ $(a \in S)$ indexed on a second set S.
Equivalently $(X, \Vdash, S)$ where $x \Vdash a \equiv x \,\epsilon\, \text{ext}\,(a)$.

We use the relation overlap between subsets $D, E \subseteq X$:

$$E \between D \equiv (\exists x \in X)(x \,\epsilon\, E \;\&\; x \,\epsilon\, D)$$

Then interior and closure of $D \subseteq X$ are defined by:

$$x \,\epsilon\, \text{int}\, D \equiv \exists a (x \Vdash a \;\&\; \text{ext}\, a \subseteq D)$$

$$x \,\epsilon\, \text{cl}\, D \equiv \forall a (x \Vdash a \to \text{ext}\, a \between D)$$

Since $E \between D$ is the logical dual of $E \subseteq D \equiv (\forall x \in X)(x \,\epsilon\, E \to x \,\epsilon\, D)$, we find that

int and cl are defined by strictly dual formulas,

obtained one from another by swapping $\forall$, $\exists$ and $\to$, $\&$.

duality and symmetry in topology 2

By looking at definitions, one can see that int and cl are obtained by composing more elementary operators between $\mathcal{P}X$ and $\mathcal{P}S$. Putting:

$$x \,\epsilon\, \mathrm{ext}\, U \equiv \Diamond x \between U, \quad a \,\epsilon\, \Diamond D \equiv \mathrm{ext}\, a \between D,$$

$$x \,\epsilon\, \mathrm{rest}\, U \equiv \Diamond x \subseteq U, \quad a \,\epsilon\, \Box D \equiv \mathrm{ext}\, a \subseteq D,$$

then

$$\mathrm{int} = \mathrm{ext}\,\Box, \quad \mathrm{cl} = \mathrm{rest}\,\Diamond$$

With no conditions on $\Vdash$, the structure $(X, \Vdash, S)$ is perfectly symmetric.

So we define the operators $\mathcal{J}, \mathcal{A}$ on $\mathcal{P}S$ symmetric of $\mathrm{int}\,, \mathrm{cl}$:

$$\mathcal{J} = \Diamond\,\mathrm{rest}\,, \quad \mathcal{A} = \Box\,\mathrm{rest}\ .$$

Since $\mathrm{ext} \dashv \Box$ and $\Diamond \dashv \mathrm{rest}$ are adjunctions:

$\mathrm{int}\,, \mathcal{J}$ are reductions (contractive, monotone, idempotent)

$\mathrm{cl}\,, \mathcal{A}$ are saturations (expansive, monotone, idempotent).

Moreover, open subsets of X, i.e. fixed points for int, coincide with those of the form $\mathrm{ext}\, U$ for some $U \subseteq S$.

They form a complete lattice which is isomorphic to fixed points for $\mathcal{A}$, which are hence called formal open subsets.

All this applies dually to closed subsets.

duality and symmetry in topology 3

X set of points, $D \subseteq X$ — S set of observables, $U \subseteq S$

$\exists\forall$ *reduction* — *concrete open* $\mathrm{int}\, D = D$ — *formal closed* $U = \mathcal{J} U$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{ext}\, U \arrow[rr, dash, "\textit{symmetric}"] \arrow[d, dash, "\textit{dual}"'] \arrow[drr, leftrightarrow, "\textit{isomorphic}"] & & \Diamond D \arrow[d, dash, "\textit{dual}"] \arrow[dll, leftrightarrow, "\textit{isomorphic}"'] \\
\mathrm{rest}\, U \arrow[rr, dash, "\textit{symmetric}"'] & & \Box D
\end{tikzcd}$$

$\forall\exists$ *saturation* — $\mathrm{cl}\, D = D$ *concrete closed* — $U = \mathcal{A} U$ *formal open*

These discoveries were buried under **ideology**: excess of assumptions

`LEM` forces validity $\mathrm{cl} = -\,\mathrm{int}\,-$

`PSA` makes the second set S useless (since ΩX itself is a set, and $\Vdash = \in$).

This is sufficient to falsify the claim that the classical paradigm is just "absolute truth"

Convergence as the only mathematical module over fully logical (structural) definitions

Up to here the structure is a basic pair $(X, \Vdash, S)$ where $\Vdash$ is **any** relation.

Recall that the open subsets are of the form $\text{ext}\, U \equiv \cup_{b \epsilon U} \text{ext}\, b$.

A topological space is a set X of points and a collection of open subsets ΩX closed under arbitrary unions and finite intersections

$$(X, \Omega X) \quad \Rightarrow \quad (X, \epsilon, \Omega X) \quad \Rightarrow \quad (X, \Vdash, S)$$

where $\text{ext}\, a \subseteq X (a \in S)$ is a base for ΩX and $x \Vdash a \equiv x \,\epsilon\, \text{ext}\, a$.

Open subsets form a topology ΩX iff $\text{ext}\,(a) \subseteq X \; (a \in S)$ is a base, that is, satisfies convergence:

B1: $\text{ext}\, a \cap \text{ext}\, b = \text{ext}\,(a \downarrow b)$

where $c \,\epsilon\, a \downarrow b \equiv \text{ext}\, c \subseteq \text{ext}\, a \;\&\; \text{ext}\, c \subseteq \text{ext}\, b$

every two neighbourhoods have a common refinement

B2: $\text{ext}\, S = X$

every point has a neighbourhood

Equivalents of convergence

In a basic pair $\mathcal{X}$, a subset of points $D \subseteq X$ is said to be convergent if

D is strongly inhabited: $D \between \text{ext}\, a$ for some a,

D is filtering: $D \between \text{ext}\, a \;\&\; D \between \text{ext}\, b \rightarrow D \between \text{ext}\,(a \downarrow b)$ for all a, b.

A subset $\alpha \subseteq S$ is said to be ideal point of $\mathcal{X}$ if

α is inhabited,

α is filtering: $D \between \text{ext}\, a \;\&\; D \between \text{ext}\, b \rightarrow D \between \text{ext}\,(a \downarrow b)$ for all a, b

α is formal closed

For every basic pair, t.f.a.e.:
- B1: $\text{ext}\, a \cap \text{ext}\, b = \text{ext}\,(a \downarrow b)$ and B2: $\text{ext}\, S = X$, so $\mathcal{X}$ is a concrete space
- every singleton in X is convergent, written $\mathcal{P}_1 X \subseteq Conv(\mathcal{X})$
- every $\Diamond x$ is an ideal point of $\mathcal{X}$

Open problem: compare this with the definition of effective topological space

continuity as a commutative square

The presence of bases S, T allows one to discover that a function $f : X \to Y$ is continuous from $(X, \Vdash, S)$ into $(Y, \Vdash', T)$ iff there is a relation s between S and T s.t. $\Vdash' \circ f = s \circ \Vdash$.

By symmetry it is natural to consider a relation r also between X and Y. Then the following conditions are equivalent:

1. r is continuous, that is $r\,x \between \text{ext}\, b \to \exists a(x \Vdash a\ \&\ \text{ext}\, a \subseteq r^{-}\, \text{ext}\, b)$
2. r^{-} is open,
3. $r^{-}\, \text{ext}\, b = \text{ext}\,(s_r{}^{-} b)$ for all $b \in T$, where $a\, s_r\, b \equiv \text{ext}\, a \subseteq r^{-}\, \text{ext}\, b$
4. there exists a relation $s : S \to T$ such that $r^{-}\, \text{ext}\, b = \text{ext}\, s^{-} b$ for all $b \in T$.

In other terms, continuity becomes $\Vdash' \circ r = s \circ \Vdash$, that is a commutative square of relations between sets:

$$
\begin{tikzcd}
X \arrow[r, "\Vdash"] \arrow[d, "r"'] & S \arrow[d, "s"] \\
Y \arrow[r, "\Vdash'"] & T
\end{tikzcd}
$$

categories **BP** and **CSpa**

BP: basic pairs $(X, \Vdash, S)$ and relation-pairs (r, s) (commutative squares)

CSpa: concrete spaces = convergent basic pairs, that is: B1-B2 hold, or equivalently every $\mathrm{cl}\,\{x\}$ is convergent

relation-pairs (r, s) preserving convergence:

r maps convergent subsets into convergent subsets,
s maps ideal points into ideal points,
r^- respects finite intersections
$r^- \,\mathrm{ext}\,(b \downarrow_{\mathcal{Y}} c) = \mathrm{ext}\,(s^- b \downarrow_{\mathcal{X}} s^- c)$, for all $b, c \in T$, and $r^- \,\mathrm{ext}\, T = \mathrm{ext}\, S$,

Pointfree topologies

Why pointfree topology? In many cases, points do not form a set. So we must obtain them as ideal points over an effective, pointfree structure
(as in the example: intervals with rational end-points as a base for the topology on real numbers; here real numbers are ideal points)

Basic topology: axiomatization of the structure deposited by a basic pair $(X, \Vdash, S)$ on S:

$(S, \mathcal{A}, \mathcal{J})$ with:

$\mathcal{A}$ saturation: $U \subseteq \mathcal{A}V \leftrightarrow \mathcal{A}U \subseteq \mathcal{A}V$

$\mathcal{J}$ reduction: $\mathcal{J}U \subseteq V \leftrightarrow \mathcal{J}U \subseteq \mathcal{J}V$

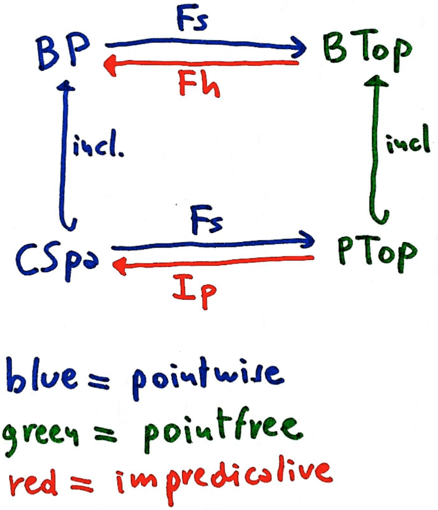
$\mathcal{A}/\mathcal{J}$ compatibility: $\mathcal{A}U \between \mathcal{J}V \leftrightarrow U \between \mathcal{J}V$

$\mathcal{J}$ is **new**: Z formal closed $= \mathcal{J}$-reduced
primitive treatment of closed subsets, also in a pointfree setting

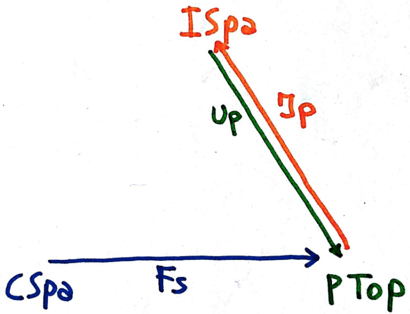
Positive topology: add convergence: $\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U{\downarrow}V)$

pointfree topology is more general than topology with points

The well known adjunction between topological spaces **Top** and locales **Loc** becomes an embedding of the category **CSpa** of concrete spaces into the category **PTop** of positive topologies



ideal spaces



for every positive topology $\mathcal{S}$, put $Ip(\mathcal{S}) \equiv (IPt(\mathcal{S}), \ni, \mathcal{S})$ ideal space

for every formal map $s : \mathcal{S} \to \mathcal{T}$, put

$$\begin{tikzcd} IPt(\mathcal{S}) \arrow[r, "\ni"] \arrow[d, "s_\exists"'] & \mathcal{S} \arrow[d, "s"'] & & \mathcal{S} \arrow[d, "s"] \arrow[ll, squiggly, shorten <=2em, shorten >=2em] \\ IPt(\mathcal{T}) \arrow[r, "\ni"'] & \mathcal{T} & & \mathcal{T} \end{tikzcd}$$

ISpa category if ideal spaces = the "image" of **PTop** under *Ip*
Conversely, *Up* forgets all what *Ip* added.

To construct a positive topology we use inductive and coinductive definitions:

cover $\lhd$, where $a \lhd U \equiv a \,\epsilon\, \mathcal{A}U$, by induction

positivity $\ltimes$, where $a \ltimes U \equiv a \,\epsilon\, \mathcal{J}U$, by coinduction

$\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U{\downarrow}V)$ is obtained if we have an operation $\downarrow$ or $\circ$ before generation.

Baire space

we obtain choice sequences as ideal points over Baire positive topology

$(\mathbb{N}^*, \lhd, \ltimes)$ positive topology

$$\frac{k \,\epsilon\, U}{k \lhd U} \qquad \frac{k * N \lhd U}{k \lhd U} \qquad \frac{l \prec k \quad l \lhd U}{k \lhd U}$$

generated by induction

$$\frac{k \ltimes U}{k \,\epsilon\, U} \qquad \frac{k \ltimes U}{k * N \ltimes U} \qquad \frac{k \prec l \quad l \ltimes U}{k \ltimes U}$$

generated by coinduction

spread = inhabited subset which is a fixed point for $\ltimes$ (formal closed)

ideal point (in any positive topology) = convergent, formal closed subset

α inhabited, $a, b \,\epsilon\, \alpha \to a{\downarrow}b \between \alpha$, and α is formal closed (hence splits the cover).

Ideal points of Baire positive topology on $\mathbb{N}^*$ coincide with functions from $\mathbb{N}$ to $\mathbb{N}$, i.e. choice sequences.

Baire space

choice sequence in a given spread = ideal point α contained (as a subset) in a formal closed subset

assuming AC! amounts to: every sequence is lawlike (as in Bishop)

absence of AC! shows that Brouwer with Bishop were talking about two different notions, and thus can reconcile them.

we wish spatial intuition **to live together with** computational interpretation

notion of choice sequence: very fragile, depends on foundational choices

Spatial intuition and computational interpretation reconciled

Bar Induction is just an equivalent formulation of spatiality of Baire positive topology:

$$\texttt{BI} \qquad \forall\alpha(k \epsilon \alpha \to U \between \alpha) \to k \lhd U$$

So BI is a specific example of a general property.

The dual to spatiality, reducibility, should be valid in Baire positive topology.

In fact, it says that every spread is inhabited by a choice sequence:

$$\texttt{SH} \qquad k \ltimes U \to \exists\alpha(k \epsilon \alpha \ \& \ \alpha \subseteq U)$$

Both BI and SH are perfectly precise and clear mathematical statements.

Spatial intuition and computational interpretation reconciled

Both $\mathtt{BI}$ and $\mathtt{SH}$ are intuitively obvious, but from the perspective of MF and positive topology they look as unprovable.

Way out: prove meta-mathematically that such ideal principles are conservative over real, pointfree topology.

Trying to prove conservativity of $\mathtt{BI}$ and $\mathtt{SH}$ is work in progress.

NB: the usual problem due to Kleene's $\mathtt{BI} + \mathtt{AC!} + \mathtt{CT} \vdash \perp$ is resolved by absence of $\mathtt{AC!}$.

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