Duality of upper and lower powerlocales on locally compact locales

Tatsuji Kawai
University of Padova

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Why powerlocales?

- Compact and overt
- Duality
- Basic Picture
- Semantics: modal logic, non-determinism

Background

Theorem (Hyland 1983)

A locale X is locally compact if and only if the exponential $\mathbb{S}^X$ over Sierpinski locale $\mathbb{S}$ exists.

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$\mathbf{LKLoc}$: the category of locally compact locales.

Corollary

There is an adjunction

$$\mathbf{LKLoc} \underset{\mathbb{S}^{(-)}}{\overset{\mathbb{S}^{(-)}}{\rightleftarrows}} \mathbf{LKLoc}^{\mathrm{op}}, \quad \bot$$

induced by the natural isomorphism

$$\begin{aligned} \mathbf{LKLoc}(X, \mathbb{S}^Y) &\cong \mathbf{LKLoc}(X \times Y, \mathbb{S}) \\ &\cong \mathbf{LKLoc}(Y \times X, \mathbb{S}) \\ &\cong \mathbf{LKLoc}(Y, \mathbb{S}^X). \end{aligned}$$

Background

Question

What is a monad on **LKLoc** induced by the adjunction?

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Theorem (Vickers 2004)

$\mathbb{S}^{\mathbb{S}^X} \cong \mathrm{P_U P_L} X \cong \mathrm{P_L P_U} X$

$\mathrm{P_L}$: the **lower powerlocale** monad.

$\mathrm{P_U}$: the **upper powerlocale** monad.

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Proposition (de Brecht & K 2016)

$\mathrm{P_U}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_L}X}$ & $\mathrm{P_L}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_U}X}$.

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$\mathrm{P_U}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_L} X}$ & $\mathrm{P_L}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_U} X}$. *So what?*

Locales

Definition

A **frame** is a poset with arbitrary joins and finite meets that distributes over joins. A **frame homomorphism** is a function that preserves finite meets and all joins. The category $\mathbf{Loc}$ of **locales** is the opposite of the category of frames.

Notations

X, Y : locales.

ΩX : the frame corresponding to a locale X.

$\Omega Y \xrightarrow{\Omega f} \Omega X$: the frame homomorphism corresponding to a locale map $f : X \to Y$.

Lower and Upper Powerlocales

Definition

A **suplattice** is a poset with arbitrary joins. A **suplattice homomorphism** is a function that preserves all joins. Write $\mathbf{SupLat}$ for the category of suplattices.

The forgetful functor $U\colon \mathbf{Frm} \to \mathbf{SupLat}$ has a left adjoint $F\colon \mathbf{SupLat} \to \mathbf{Frm}$:

- for each suplattice D, there exists a frame $F(D)$ and a suplattice homomorphism $\iota^L_D\colon D \to F(D)$,
- for any frame Y and a suplattice homomorphism $f\colon D \to Y$, there exists a unique frame homomorphism $\bar{f}\colon F(D) \to Y$ such that

$$\begin{array}{ccc} F(D) & \overset{\bar{f}}{\dashrightarrow} & Y. \\ {\scriptstyle \iota^L_D}\uparrow & \nearrow_{f} & \\ D & & \end{array}$$

Lower powerlocales

Definition

Let X be a locale. The **lower powerlocale** $\mathrm{P_L}X$ is the locale corresponding to the frame $F(U(\Omega X))$.

The lower powerlocales form a monad $\langle \mathrm{P_L}, \eta^L, \mu^L \rangle$, where η^L_X and μ^L_X are given by

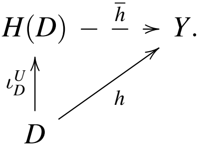
$$\begin{array}{ccc} \Omega\mathrm{P_L}X & \xrightarrow{\Omega\eta^L_X} & \Omega X \\ {\scriptstyle \iota^L_X}\uparrow & \nearrow{\scriptstyle \mathrm{id}_{\Omega X}} & \\ \Omega X & & \end{array} \qquad \begin{array}{ccc} \Omega\mathrm{P_L}X & \xrightarrow{\Omega\mu^L_X} & \Omega\mathrm{P_L}\mathrm{P_L}X \\ {\scriptstyle \iota^L_X}\uparrow & & \uparrow{\scriptstyle \iota^L_{\mathrm{P_L}X}} \\ \Omega X & \xrightarrow{\iota^L_X} & \Omega\mathrm{P_L}X. \end{array}$$

Definition

A **preframe** is a poset with directed joins and finite meets which distributes over directed joins. A **preframe homomorphism** is a function that preserves finite meets and directed joins. Write $\mathbf{PrFrm}$ for the category of preframes.

The forgetful functor $U\colon \mathbf{Frm} \to \mathbf{PrFrm}$ has a left adjoint $H\colon \mathbf{PrFrm} \to \mathbf{Frm}$:

- for each preframe D, there exists a frame $H(D)$ and a preframe homomorphism $\iota_D^U\colon D \to H(D)$,
- for any frame Y and a preframe homomorphism $h\colon D \to Y$, there exists a unique frame homomorphism $\overline{h}\colon H(D) \to Y$ such that



Upper powerlocales

Definition

Let X be a locale. The **upper powerlocale** $\mathrm{P_U}X$ is the locale corresponding to the frame $H(U(\Omega X))$.

The upper powerlocales form a monad $\langle \mathrm{P_U}, \eta^U, \mu^U \rangle$, where η^U_X and μ^U_X are given by

$$\begin{array}{ccc} \Omega \mathrm{P_U} X & \xrightarrow{\Omega \eta^U_X} & \Omega X \\ \uparrow{\scriptstyle \iota^U_X} & \nearrow{\scriptstyle \mathrm{id}_{\Omega X}} & \\ \Omega X & & \end{array} \qquad \begin{array}{ccc} \Omega \mathrm{P_U} X & \xrightarrow{\Omega \mu^U_X} & \Omega \mathrm{P_U} \mathrm{P_U} X \\ \uparrow{\scriptstyle \iota^U_X} & & \uparrow{\scriptstyle \iota^U_{\mathrm{P_U} X}} \\ \Omega X & \xrightarrow{\iota^U_X} & \Omega \mathrm{P_U} X. \end{array}$$

Order Enrichment and Distributivity

Order enrichments

Definition

The category of **Poset** of posets is a poset enriched category (i.e. homesets are posets), where morphisms are ordered pointwise.
Loc is poset-enriched by **specialization order** given by

$$
\begin{aligned}
f \leq g &\overset{\text{def}}{\iff} \Omega f \leq_{\mathbf{Poset}} \Omega g \\
&\overset{\text{def}}{\iff} (\forall y \in Y)\, \Omega f(y) \leq \Omega g(y).
\end{aligned}
$$

Definition

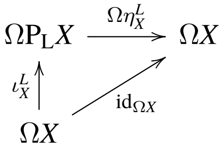
In a poset enriched category $\mathbb{C}$, a morphism $f\colon X \to Y$ is **the left adjoint** to $g\colon Y \to X$, written $f \dashv g$, if $\mathrm{id}_X \leq g \circ f \;\;\&\;\; f \circ g \leq \mathrm{id}_Y$.

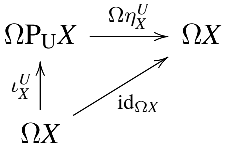
Order enrichments

Lemma

For any locale X, we have (in $\mathbf{Poset}$*)*

- $\iota^L_X \dashv \Omega\eta^L_X \quad (\iff \iota^L_X \circ \Omega\eta^L_X \leq \mathrm{id}_{\Omega \mathrm{P_L} X})$;
- $\Omega\eta^U_X \dashv \iota^U_X \quad (\iff \mathrm{id}_{\Omega \mathrm{P_U} X} \leq \iota^U_X \circ \Omega\eta^U_X)$.





KZ-monads

Definition

Let $\langle T, \eta, \mu \rangle$ be a monad on a poset enriched category $\mathbb{C}$, where T preserves the order on morphisms. Then, T is a **KZ-monad (coKZ-monad)** if $T\eta_X \leq \eta_{TX}$ ($\eta_{TX} \leq T\eta_X$).

Proposition

$\langle \mathrm{P_L}, \eta^L, \mu^L \rangle$ *is a KZ-monad and* $\langle \mathrm{P_L}, \eta^U, \mu^U \rangle$ *is a coKZ-monad.*

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Proposition

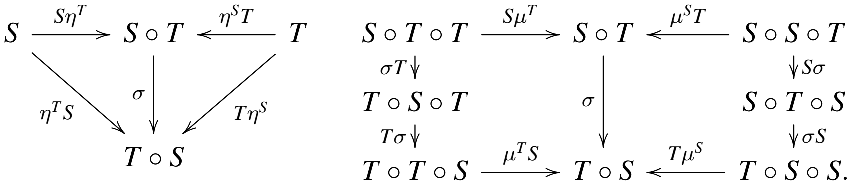
Let $\langle T, \eta, \mu \rangle$ be a KZ-monad on a poset enriched category $\mathbb{C}$. Then, the following are equivalent.

1. *$\alpha \colon TX \to X$ is a T-algebra;*
2. *$\alpha \dashv \eta_X$ & $\alpha \circ \eta_X = \mathrm{id}_X$;*
3. *$\alpha \circ \eta_X = \mathrm{id}_X$.*

In particular, T-algebra structure on X (if it exists) is unique.

Distributivity

Let $\langle T, \eta^T, \mu^T \rangle$ and $\langle S, \eta^S, \mu^S \rangle$ be monads. A **distributive law** of T over S is a natural transformation $\sigma \colon S \circ T \to T \circ S$ which makes the diagrams commutes.



Then, $T \circ S$ is a monad with

$$\eta = \mathrm{id} \xrightarrow{\eta^S} S \xrightarrow{\eta^T S} T \circ S,$$

$$\mu = T \circ S \circ T \circ S \xrightarrow{T\sigma S} T \circ T \circ S \circ S \xrightarrow{\mu^T S \circ S} T \circ S \circ S \xrightarrow{T\mu^S} T \circ S.$$

Distributivity

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$$\begin{array}{ccccc}
S & \xrightarrow{S\eta^T} & S \circ T & \xleftarrow{\eta^S T} & T \\
& {\scriptstyle \eta^T S} \searrow & \downarrow {\scriptstyle \sigma} & \swarrow {\scriptstyle T\eta^S} & \\
& & T \circ S & &
\end{array}
\qquad
\begin{array}{ccccc}
S \circ T \circ T & \xrightarrow{S\mu^T} & S \circ T & \xleftarrow{\mu^S T} & S \circ S \circ T \\
{\scriptstyle \sigma T} \downarrow & & & & \downarrow {\scriptstyle S\sigma} \\
T \circ S \circ T & & \downarrow {\scriptstyle \sigma} & & S \circ T \circ S \\
{\scriptstyle T\sigma} \downarrow & & & & \downarrow {\scriptstyle \sigma S} \\
T \circ T \circ S & \xrightarrow{\mu^T S} & T \circ S & \xleftarrow{T\mu^S} & T \circ S \circ S.
\end{array}$$

Then, $T \circ S$ is a monad with

$$\eta = \mathrm{id} \xrightarrow{\eta^S} S \xrightarrow{\eta^T S} T \circ S,$$

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Proposition (Vickers 2004)

There is a natural isomorphism $\mathrm{P_L} \circ \mathrm{P_U} \cong \mathrm{P_U} \circ \mathrm{P_L}$ *which (together with its inverse) is a distributive law of* $\mathrm{P_L}$ *over* $\mathrm{P_U}$ *and vice versa.*

Double powerlocales

Definition

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Lemma (Vickers 2004)

Every $\mathbb{P}$-algebra is also $\mathrm{P_L}$-algebra and $\mathrm{P_U}$-algebra. Moreover, $\mathbb{P}$-algebra structure on a object X (if it exists) is unique.

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Proof. If $\mathbb{P}X \xrightarrow{\alpha} X$ is an $\mathbb{P}$-algebra, its $\mathrm{P_L}$-algebra structure is

$$\mathrm{P_L}X \xrightarrow{\mathrm{P_L}\eta^U_X} \mathrm{P_U}\mathrm{P_L}X \cong \mathbb{P}X \xrightarrow{\alpha} X,$$

which is a retract of $\eta^L_X \colon X \to \mathrm{P_L}X$ (note: $\mathrm{P_L}$ is a KZ-monad). $\square$

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$$\mathbb{P}\mathrm{P_U}X \cong \mathrm{P_L}{\mathrm{P_U}}^2X \xrightarrow{\mathrm{P_L}\mu^U_X} \mathrm{P_L}\mathrm{P_U}X \cong \mathbf{P_U P_L}\boldsymbol{X} \xrightarrow{\mathbf{P_U}\boldsymbol{\alpha}} \mathbf{P_U}\boldsymbol{X}$$

  is a $\mathbb{P}$-algebra and $\eta^U_X : X \to \mathrm{P_U}X$ is a $\mathrm{P_L}$-algebra morphism;

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- for any $\mathbb{P}$-algebra $\mathbb{P}Y \xrightarrow{\beta} Y$ and $\mathrm{P_L}$-algebra morphism $f : X \to Y$, there is a unique $\mathbb{P}$-algebra morphism $\bar{f} : \mathrm{P_U}X \to Y$ such that

$$\begin{array}{ccc} \mathrm{P_U}X & \xrightarrow{\mathrm{P_U}f} & \mathrm{P_U}Y \\ {\scriptstyle \eta^U_X}\uparrow & \overset{\bar{f}}{\searrow} & \downarrow{\scriptstyle \beta_L} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

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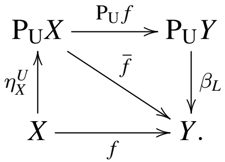

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KZ-comonads

The adjunctions $\mathrm{P_L}\text{-}\mathbf{Alg} \underset{\perp}{\overset{\mathrm{P_U}}{\rightleftarrows}} \mathbb{P}\text{-}\mathbf{Alg}$ and $\mathrm{P_U}\text{-}\mathbf{Alg} \underset{\perp}{\overset{\mathrm{P_L}}{\rightleftarrows}} \mathbb{P}\text{-}\mathbf{Alg}$ induce comonads on $\mathbb{P}\text{-}\mathbf{Alg}$, denoted by $\widetilde{\mathrm{P_U}}$ and $\widetilde{\mathrm{P_L}}$ respectively.

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Definition

Let $\langle T, \varepsilon, \delta \rangle$ be a comonad on a poset enriched category $\mathbb{C}$, where T preserves the order on morphisms. Then, T is a ***KZ*-comonad (co*KZ*-comonad)** if $T\varepsilon_X \leq \varepsilon_{TX}$ (resp. $\varepsilon_{TX} \leq T\varepsilon_X$).

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Proposition

$\widetilde{\mathrm{P_U}}$ is a KZ-comonad and $\widetilde{\mathrm{P_L}}$ is a coKZ-comonad on R-$\mathbf{Alg}$.

$\mathbb{P}$-$\mathbf{Alg}$ morphism

$$\begin{array}{ccc} \mathbb{P}X & \xrightarrow{\mathbb{P}h} & \mathbb{P}\mathrm{P_U}X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\widetilde{\mathrm{P_U}}\alpha} \\ X & \xrightarrow{h} & \mathrm{P_U}X \end{array}$$

is a $\widetilde{\mathrm{P_U}}$-coalgebra structure on

$\mathbb{P}X \xrightarrow{\alpha} X \iff h$ is a section of $\varepsilon^{\widetilde{U}}_{\alpha} \colon \mathrm{P_U}X \to X$.

Scott topologies

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Proposition

Let X be a locally compact locale (i.e. a locale s.t. $\mathbb{S}^X$ exists). Then

- $\Omega\mathbb{S}^X$ *is a Scott topology on* ΩX*;*
- ΩX *is a continuous lattice.*

Thus, the following are equivalent for $X, Y \in \mathbf{LKLoc}$*:*

- *a locale morphism* $f\colon \mathbb{S}^X \to \mathbb{S}^Y$*;*
- *a Scott continuous map* $h\colon \Omega X \to \Omega Y$*;*
- *a function* $h\colon \Omega X \to \Omega Y$ *that preserves directed joins.*

For a Scott continuous $h\colon \Omega X \to \Omega Y$, write $\Sigma h\colon \mathbb{S}^X \to \mathbb{S}^Y$ for the corresponding locale morphism.

Embedding into $\mathbb{P}$-$\mathbf{Alg}$

Proposition

The assignment $X \mapsto \mathbb{S}^X$ induces an embedding $\mathbf{LKLoc}^{\mathrm{op}} \to \mathbb{P}$-$\mathbf{Alg}$. In particular, $\mathbb{S}^X$ has a unique $\mathbb{P}$-algebra structure.

For each locale map $f\colon X \to Y$ in $\mathbf{LKLoc}$, we have $\mathbb{S}^f = \Sigma\Omega f$.

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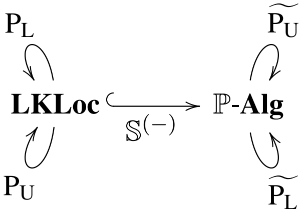

Main Results

The first duality $\mathrm{P_U}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_L}X}$

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If X locally compact, there is a natural isomorphism $\mathrm{P_U}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_L}X}$.

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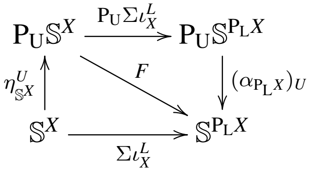

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1. $F\colon \mathrm{P_U}\mathbb{S}^X \to \mathbb{S}^{\mathrm{P_L}X}$ is defined by

$$\begin{array}{ccc}
\mathrm{P_U}\mathbb{S}^X & \xrightarrow{\mathrm{P_U}\Sigma\iota^L_X} & \mathrm{P_U}\mathbb{S}^{\mathrm{P_L}X} \\
\uparrow {\scriptstyle \eta^U_{\mathbb{S}X}} & \overset{F}{\searrow} & \downarrow {\scriptstyle (\alpha_{\mathrm{P_L}X})_U} \\
\mathbb{S}^X & \xrightarrow[\Sigma\iota^L_X]{} & \mathbb{S}^{\mathrm{P_L}X}
\end{array}$$

2. $G\colon \mathbb{S}^{\mathrm{P_L}X} \to \mathrm{P_U}\mathbb{S}^X$ corresponds to a preframe morphism $g\colon \Omega\mathbb{S}^X \to \Omega\mathbb{S}^{\mathrm{P_L}X}$. If G were an inverse of F, we must have $\Omega\Sigma\iota^L_X \dashv g$. Since $\Omega\Sigma\iota^L_X \dashv g \iff \mathrm{P_U}\mu^L_X \cong \Sigma\Omega\Sigma\iota^L_X \dashv \Sigma g$, the right adjoint Σg corresponds to $\mathrm{P_U}\eta^L_{\mathrm{P_L}X}$ by

$$\begin{array}{ccc}
\mathbb{S}^{\mathbb{S}^X} & \xrightarrow{\Sigma g} & \mathbb{S}^{\mathbb{S}^{\mathrm{P_L}X}} \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \cong} \\
\mathrm{P_U}\mathrm{P_L}X & \xrightarrow{\mathrm{P_U}\eta^L_{\mathrm{P_L}X}} & \mathrm{P_U}\mathrm{P_L}\mathrm{P_L}X.
\end{array}$$

Suplattice and $\mathrm{P_L}$-algebra homomorphisms

Theorem

For any locally compact locales X, Y, there is a natural isomorphisms

$$\mathrm{P_L}\text{-}\mathbf{Alg}(\mathbb{S}^X, \mathbb{S}^Y) \cong \mathbf{SupLat}(\Omega X, \Omega Y).$$

Proof.

$$\begin{aligned}
\mathbf{SupLat}(\Omega X, \Omega Y) &\cong \mathbf{LKLoc}(Y, \mathrm{P_L} X) \\
&\cong \mathbb{P}\text{-}\mathbf{Alg}(\mathbb{S}^{\mathrm{P_L} X}, \mathbb{S}^Y) \\
&\cong \mathbb{P}\text{-}\mathbf{Alg}(\mathrm{P_U}\mathbb{S}^X, \mathbb{S}^Y) \\
&\cong \mathrm{P_L}\text{-}\mathbf{Alg}(\mathbb{S}^X, \mathbb{S}^Y).
\end{aligned}$$

□

$\mathrm{P_L}$-algebras and $\widetilde{\mathrm{P_U}}$-coalgebras

From the diagram
$$\begin{array}{ccc} \mathrm{P_U}\mathbb{S}^X & \xrightarrow{F} & \mathbb{S}^{\mathrm{P_L}X} \\ {\scriptstyle \eta^U_{\mathbb{S}^X}}\uparrow & \nearrow{\scriptstyle \Sigma\iota^L_X} & \\ \mathbb{S}^X & & \end{array}$$
we get
$$\begin{array}{ccc} \mathrm{P_U}\mathbb{S}^X & \xrightarrow{F} & \mathbb{S}^{\mathrm{P_L}X} \\ {\scriptstyle \eta^U_{\mathbb{S}^X}}\uparrow & \swarrow{\scriptstyle \mathbb{S}^{\eta^L_X}} & \\ \mathbb{S}^X \circlearrowleft {\scriptstyle \mathrm{id}} & & \end{array}.$$

Lemma

The composite $\mathbb{S}^{\eta^L_X} \circ F$ *is the counit* $\varepsilon^{\widetilde{U}}_{\mathbb{S}^X}$ *of* $\widetilde{\mathrm{P_U}}$.

$\mathrm{P_L}$-algebras and $\widetilde{\mathrm{P_U}}$-coalgebras

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Lemma

The composite $\mathbb{S}^{\eta^L_X} \circ F$ *is the counit* $\varepsilon^{\widetilde{U}}_{\mathbb{S}^X}$ *of* $\widetilde{\mathrm{P_U}}$*.*

Proof. By definition, the counit of the comonad $\widetilde{\mathrm{P_U}}$ satisfies

$$\begin{array}{ccc} \mathrm{P_U}\mathbb{S}^X & \xrightarrow{\varepsilon^{\widetilde{U}}_{\mathbb{S}^X}} & \mathbb{S}^X. \\ {\scriptstyle \eta^U_{\mathbb{S}X}}\uparrow & \nearrow{\scriptstyle \mathrm{id}_{\mathbb{S}^X}} & \\ \mathbb{S}^X & & \end{array}$$

□

$\mathrm{P_L}$-algebras and $\widetilde{\mathrm{P_U}}$-coalgebras

$\mathrm{P_L}$-$\mathbf{Alg_{LK}}$: the category of $\mathrm{P_L}$-algebras on $\mathbf{LKLoc}$.

$\widetilde{\mathrm{P_U}}$-$\mathbf{coAlg}$: the category of $\widetilde{\mathrm{P_U}}$-coalgebras on $\mathbb{P}$-$\mathbf{Alg}$.

Theorem

The embedding $\mathbf{LKLoc}^{\mathrm{op}} \xrightarrow{\mathbb{S}^{(-)}} \mathbb{P}\text{-}\mathbf{Alg}$ *restricts to an embedding* $\mathrm{P_L}\text{-}\mathbf{Alg_{LK}} \to \widetilde{\mathrm{P_U}}\text{-}\mathbf{coAlg}$.

$\mathrm{P_L}$-algebras and $\widetilde{\mathrm{P_U}}$-coalgebras

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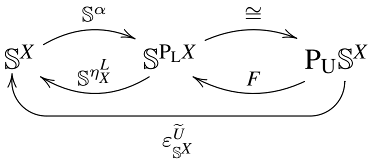
Proof. If $\mathrm{P_L}X \xrightarrow{\alpha} X$ is a $\mathrm{P_L}$-algebra, then $\mathbb{S}^X \xrightarrow{\mathbb{S}^\alpha} \mathbb{S}^{\mathrm{P_L}X} \cong \mathrm{P_U}\mathbb{S}^X$ is a $\widetilde{\mathrm{P_U}}$-coalgebra structure on $\mathbb{S}^X$:



□

The second duality $\mathrm{P_L}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_U}X}$

Proposition

If X locally compact, there is a natural isomorphism $\mathrm{P_L}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_U}X}$.

The second duality $\mathrm{P_L}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_U}X}$

Proposition

If X locally compact, there is a natural isomorphism $\mathrm{P_L}\mathbb{S}^X \cong \mathbb{S}^{\mathrm{P_U}X}$.

Proof. We have natural isomorphisms:

$$\mathbb{S}^{\mathbf{P_L}\mathbb{S}^X} \cong \mathbf{P_U}\mathbb{S}^{\mathbb{S}^X} \cong \mathrm{P_U}\mathrm{P_L}\mathrm{P_U}X \cong \mathrm{P_L}\mathrm{P_U}\mathrm{P_U}X \cong \mathbb{S}^{\mathbb{S}^{\mathrm{P_U}X}}.$$

Since $\mathbf{LKLoc}^{\mathrm{op}} \xrightarrow{\mathbb{S}^{(-)}} \mathbb{P}\text{-}\mathbf{Alg}$ is an embedding, we have an isomorphism $H\colon \mathrm{P_L}\mathbb{S}^X \xrightarrow{\cong} \mathbb{S}^{\mathrm{P_U}X}$. □

Theorem

For any locally compact locales X, Y, there is a natural isomorphisms $\mathrm{P_U}\text{-}\mathbf{Alg}(\mathbb{S}^X, \mathbb{S}^Y) \cong \mathbf{PrFrm}(\Omega X, \Omega Y)$.

$\mathrm{P_U}$-algebras and $\widetilde{\mathrm{P_L}}$-coalgebras

Lemma

The diagram commutes:

$$\begin{array}{ccc} \mathrm{P_L}\mathbb{S}^X & \xrightarrow{\;H\;} & \mathbb{S}^{\mathrm{P_U}X}. \\ {\scriptstyle \eta^L_{\mathbb{S}X}}\uparrow & \swarrow {\scriptstyle \mathbb{S}^{\eta^U_X}} & \\ \mathbb{S}^X \circlearrowleft {\scriptstyle \mathrm{id}} & & \end{array}$$

$\mathrm{P_U}$-algebras and $\widetilde{\mathrm{P_L}}$-coalgebras

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The diagram commutes:

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Proof. It suffices to show that

$$\begin{array}{ccc} \mathbb{S}^{\mathrm{P_L}\mathbb{S}^X} & \xleftarrow{\;\mathbb{S}^H\;} & \mathbb{S}^{\mathbb{S}^{\mathrm{P_U}X}} \\ {\scriptstyle \mathbb{S}^{\eta^L_{\mathbb{S}^X}}}\downarrow & \nearrow {\scriptstyle \mathbb{S}^{\mathbb{S}^{\eta^U_X}}} & \\ \mathbb{S}^{\mathbb{S}^X} & & \end{array}$$

commutes.

$\mathrm{P_U}$-algebras and $\widetilde{\mathrm{P_L}}$-coalgebras

Lemma

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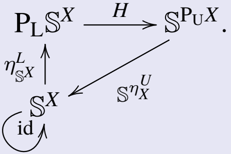
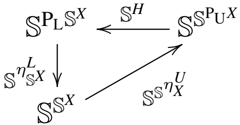
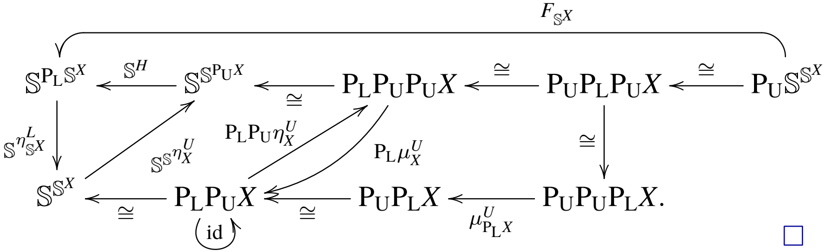


Proof. It suffices to show that



commutes.



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$\mathrm{P_U}$-algebras and $\widetilde{\mathrm{P_L}}$-coalgebras

Theorem

The embedding $\mathbf{LKLoc}^{\mathrm{op}} \xrightarrow{\mathbb{S}^{(-)}} \mathbb{P}\text{-}\mathbf{Alg}$ *restricts to an embedding* $\mathrm{P_U}\text{-}\mathbf{Alg}_{\mathbf{LK}} \to \widetilde{\mathrm{P_L}}\text{-}\mathbf{coAlg}$.

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