Computing with infinite data via proofs

Helmut Schwichtenberg, Hideki Tsuiki and Franziskus Wiesnet
(with help by Quirin Scholl)

June 28, 2017

Signed Digit Representation of Reals

Signed Digit Representation of Reals

A real number $x \in [-1, 1]$ can be wirtten as stream of signed digits

$$x = \sum_{i=1}^{\infty} \frac{d_i}{2^i} = d_1 d_2 d_3 \ldots$$

where $d_i \in \mathbf{Sd} := \{\bar{1}, 0, 1\}$. We write ${}^{co}\mathbf{I}x$ or $x \in {}^{co}\mathbf{I}$ for "x has a SD representation".

Signed Digit Representation of Reals

A real number $x \in [-1, 1]$ can be wirtten as stream of signed digits

$$x = \sum_{i=1}^{\infty} \frac{d_i}{2^i} = d_1 d_2 d_3 \dots$$

where $d_i \in \mathbf{Sd} := \{\bar{1}, 0, 1\}$. We write ${}^{co}\mathbf{I}x$ or $x \in {}^{co}\mathbf{I}$ for "x has a SD representation".

Our goals are algorithms for the arithmetic functions especially the arithmetic mean and the division.

Formalisation of the Signed Digit Representation

Formalisation of the Signed Digit Representation

Reflection: If a real number x has an SD representation, there is a signed digit d and a real number x' which also has a SD representation such that $x = \frac{d+x'}{2}$.

Formalisation of the Signed Digit Representation

Reflection: If a real number x has an SD representation, there is a signed digit d and a real number x' which also has a SD representation such that $x = \frac{d+x'}{2}$.
So we have the following elimination axiom of ${}^{co}\mathbf{I}$:

$$\forall^{nc}_{x}.{}^{co}\mathbf{I}x \to \exists_{d,x'} \left(\mathbf{Sd}\ d \wedge {}^{co}\mathbf{I}x' \wedge |x| \leq 1 \wedge x = \frac{x'+d}{2} \right)$$

Formalisation of the Signed Digit Representation

Reflection: If a real number x has an SD representation, there is a signed digit d and a real number x' which also has a SD representation such that $x = \frac{d+x'}{2}$.
So we have the following elimination axiom of ${}^{co}\mathbf{I}$:

$$\forall^{nc}_{x}.{}^{co}\mathbf{I}x \to \exists_{d,x'} \left(\mathbf{Sd}\ d \wedge {}^{co}\mathbf{I}x' \wedge |x| \leq 1 \wedge x = \frac{x'+d}{2} \right)$$

The SD representation of a real is an infinit list of signed digeds. The datatyp of a stream is therefore

```
data Sd = SdL | SdM | SdR
data Str = Sd :~: Str
```

Formalisation of the Signed Digit Representation

Reflection: If a real number x has an SD representation, there is a signed digit d and a real number x' which also has a SD representation such that $x = \frac{d+x'}{2}$.
So we have the following elimination axiom of ${}^{co}\mathbf{I}$:

$$\forall^{nc}_x . {}^{co}\mathbf{I}x \to \exists_{d,x'} \left(\mathbf{Sd}\ d \land {}^{co}\mathbf{I}x' \land |x| \leq 1 \land x = \frac{x'+d}{2} \right)$$

The SD representation of a real is an infinit list of signed digeds. The datatyp of a stream is therefore

```
data Sd = SdL | SdM | SdR
data Str = Sd :~: Str
```

Using the elimination axiom ${}^{co}\mathbf{I}$ corresponds to the application of the destructor

```
strDestr :: Str -> (Sd, Str)
strDestr (d :~: str) = (d, str)
```

If we want to prove that a real number x has a SD representation, we have to give an algorithm which provides the digeds of x.

If we want to prove that a real number x has a SD representation, we have to give an algorithm which provides the digeds of x.
The introduction axiom of ${}^{co}\mathbf{I}$ is

$$\forall^{nc}_{x}.Xx \to$$
$$\forall^{nc}_{x}\left(Xx \to \exists^{r}_{d,x'}\left(\mathbf{Sd}\ d \wedge ({}^{co}\mathbf{I}x' \vee Xx') \wedge |x| \leq 1 \wedge x = \frac{d+x'}{2}\right)\right)$$
$$\to {}^{co}\mathbf{I}x.$$

If we want to prove that a real number x has a SD representation, we have to give an algorithm which provides the digeds of x.
The introduction axiom of ${}^{co}\mathbf{I}$ is

$$\forall^{nc}_{x}.Xx \to$$
$$\forall^{nc}_{x}\left(Xx \to \exists^{r}_{d,x'}\left(\mathbf{Sd}\ d \wedge ({}^{co}\mathbf{I}x' \vee Xx') \wedge |x| \leq 1 \wedge x = \frac{d+x'}{2}\right)\right)$$
$$\to {}^{co}\mathbf{I}x.$$

```
strCoRec :: t -> (t -> (Sd, Either Str t)) -> Str
strCoRec t f = let (d, strt) = f t in d :~: case strt of
  Left str -> str
  Right t0 -> strCoRec t0 f
```

Lemma

Every real between -1 and 1 has a SD representation.

Proof.

Lemma

Every real between -1 and 1 has a SD representation.

Proof. *We use the introduction axiom of* ${}^{co}\mathbf{I}$ *with the predicate* $Xx := \exists_y(y = x \wedge -1 \leq y \leq 1)$ *and have to prove:*

$$\forall^{nc}_x . \exists_y(y = x \wedge -1 \leq y \leq 1) \to \exists^r_{d,x'} \left(\mathbf{Sd}\ d \wedge ({}^{co}\mathbf{I}x' \vee Xx') \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2} \right)$$

Lemma

Every real between -1 and 1 has a SD representation.

Proof. *We use the introduction axiom of ${}^{co}\mathbf{I}$ with the predicate $Xx := \exists_y (y = x \wedge -1 \leq y \leq 1)$ and have to prove:*

$$\forall^{nc}_x . \exists_y (y = x \wedge -1 \leq y \leq 1) \to$$
$$\exists^r_{d,x'} \left(\mathbf{Sd}\ d \wedge ({}^{co}\mathbf{I}x' \vee Xx') \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2} \right)$$

Let therefore x, y and $x = y \wedge -1 \leq y \leq 1$ be given. Define $\langle as, M \rangle := y$ then we distinguish the following three cases:

Lemma

Every real between -1 *and* 1 *has a SD representation.*

Proof. *We use the introduction axiom of* ${}^{co}\mathbf{I}$ *with the predicate* $Xx := \exists_y(y = x \wedge -1 \leq y \leq 1)$ *and have to prove:*

$$\forall^{nc}_x . \exists_y(y = x \wedge -1 \leq y \leq 1) \to \exists^r_{d,x'} \left(\mathbf{Sd}\ d \wedge ({}^{co}\mathbf{I}x' \vee Xx') \wedge |x| \leq 1 \wedge x = \frac{d + x'}{2} \right)$$

Let therefore x, y *and* $x = y \wedge -1 \leq y \leq 1$ *be given. Define* $\langle as, M \rangle := y$ *then we distinguish the following three cases:*
If $as(2) \leq -\frac{1}{4}$ *it follows* $y \leq 0$ *and therefore we define* $d := -1$ *and* $x' := 2x + 1$.
If $as(2) \geq \frac{1}{4}$ *it follows* $y \geq 0$ *and therefore we define* $d := 1$ *and* $x' := 2x - 1$.
Otherwise we get $-\frac{1}{2} \leq y \leq \frac{1}{2}$ *and define* $d := 0$ *and* $x' := 2x$. □

```
RealToStream :: Rea -> Str
RealToStream x0 = strCoRec x0 f where
  f x1@(RealConstr rs m)
    | rs (m 2) <= -1/4  = (SdL, Right (2*x1+1))
    | rs (m 2) <=  1/4  = (SdM, Right (2*x1  ))
    | otherwise         = (SdR, Right (2*x1-1))
```

Theorem

If ${}^{co}\mathbf{I}x$ *and* ${}^{co}\mathbf{I}y$*, we also have* ${}^{co}\mathbf{I}\frac{x+y}{2}$*.*

Theorem

If ${}^{co}\mathbf{I}x$ *and* ${}^{co}\mathbf{I}y$, *we also have* ${}^{co}\mathbf{I}\frac{x+y}{2}$.

Proof. *Observation: From* ${}^{co}\mathbf{I}x$ *and* ${}^{co}\mathbf{I}y$ *we get* $d, e \in \mathbf{Sd}$ *and* $x', y' \in {}^{co}\mathbf{I}$ *such that* $x = \frac{d+x'}{2}$ *and* $y = \frac{e+y'}{2}$. *It follows*

$$\frac{x+y}{2} = \frac{\frac{d+x'}{2} + \frac{e+y'}{2}}{2} = \frac{x'+y'+j}{4}$$

for some $j \in \{-2, -1, 0, 1, 2\}$.

So it is sufficient to show that

$$\mathbf{P} := \{\frac{x+y+j}{4} | x, y \in {}^{co}\mathbf{I} \wedge j \in \{-2,-1,0,1,2\}\} \subseteq {}^{co}\mathbf{I}$$

So it is sufficient to show that

$$\mathbf{P} := \{\frac{x+y+j}{4} | x, y \in {}^{co}\mathbf{I} \land j \in \{-2,-1,0,1,2\}\} \subseteq {}^{co}\mathbf{I}$$

Let $\frac{x+y+j}{4} \in \mathbf{P}$ and $d, e \in \mathbf{Sd}$, $x', y' \in {}^{co}\mathbf{I}$ with $x = \frac{d+x'}{2}$ and $y = \frac{e+y'}{2}$.

So it is sufficient to show that

$$\mathbf{P} := \{\frac{x+y+j}{4} | x, y \in {}^{co}\mathbf{I} \wedge j \in \{-2,-1,0,1,2\}\} \subseteq {}^{co}\mathbf{I}$$

Let $\frac{x+y+j}{4} \in \mathbf{P}$ and $d, e \in \mathbf{Sd}$, $x', y' \in {}^{co}\mathbf{I}$ with $x = \frac{d+x'}{2}$ and $y = \frac{e+y'}{2}$.

We define $K : \{-6, \dots, 6\} \to \mathbf{Sd}$ and $J : \{-6, \dots, 6\} \to \{-2,-1,0,1,2\}$ such that $4K(a) + J(a) = a$ for all $a \in \{-6, \dots, 6\}$.

So it is sufficient to show that

$$\mathbf{P} := \{\frac{x+y+j}{4} | x, y \in {}^{co}\mathbf{I} \wedge j \in \{-2,-1,0,1,2\}\} \subseteq {}^{co}\mathbf{I}$$

Let $\frac{x+y+j}{4} \in \mathbf{P}$ and $d, e \in \mathbf{Sd}$, $x', y' \in {}^{co}\mathbf{I}$ with $x = \frac{d+x'}{2}$ and $y = \frac{e+y'}{2}$.

We define $K : \{-6,\dots,6\} \to \mathbf{Sd}$ and $J : \{-6,\dots,6\} \to \{-2,-1,0,1,2\}$ such that $4K(a) + J(a) = a$ for all $a \in \{-6,\dots,6\}$. Then we have:

$$\frac{x+y+j}{4} = \frac{\frac{d+x'}{2} + \frac{e+y'}{2} + j}{4} = \frac{d+e+2j+x'+y'}{8} =$$

$$\frac{4K(d+e+2j) + J(d+e+2j) + x' + y'}{8}$$

$$= \frac{K(d+e+2j) + \frac{J(d+e+2j)+x'+y'}{4}}{2}$$

□

```
k n | n >  2   =  1
    | n < -2   = -1
    | otherwise = 0

j 6 =  2  j 5 =  1  j 4 =  0  j 3 = -1  j 2 =  2
j 1 =  1  j 0 =  0  j (-1) = -1
j (-2) = -2  j (-3) = 1  j (-4) = 0  j (-5) = -1  j (-6) = -2

step :: (Int, Str, Str) -> (Sd, Either Str (Int, Str, Str))
step (t, d :~: u, e :~: v) =
    (toEnum (k num), Right (j num, u, v))
     where num = fromEnum d + fromEnum e + 2*t

cCoIAverage :: Str -> Str -> Str
cCoIAverage (d :~: u) (e :~: v) =
            strCoRec (fromEnum d + fromEnum e, u, v) step
```

Lemma

$$\forall_x^{nc}.{}^{co}\mathbf{I}x \to |x| \leq \frac{1}{2} \to {}^{co}\mathbf{I}(2x)$$

Lemma

$$\forall^{nc}_x . {}^{co}\mathbf{I}x \to |x| \leq \frac{1}{2} \to {}^{co}\mathbf{I}(2x)$$

Proof. *Because of* ${}^{co}\mathbf{I}x$ *we have* $x' \in {}^{co}\mathbf{I}$ *and* $d \in \mathbf{Sd}$ *such that* $x = \frac{d+x'}{2}$*.*
case differentiation by d*:*

Lemma

$$\forall_x^{nc}.{}^{co}\mathbf{I}x \to |x| \leq \frac{1}{2} \to {}^{co}\mathbf{I}(2x)$$

Proof. *Because of* ${}^{co}\mathbf{I}x$ *we have* $x' \in {}^{co}\mathbf{I}$ *and* $d \in \mathbf{Sd}$ *such that* $x = \frac{d+x'}{2}$.
case differentiation by d*:*
If $d = 0$ *we are done, because then* $2x = x' \in {}^{co}\mathbf{I}$.

Lemma

$$\forall^{nc}_x . {}^{co}\mathbf{I}x \to |x| \leq \frac{1}{2} \to {}^{co}\mathbf{I}(2x)$$

Proof. *Because of* ${}^{co}\mathbf{I}x$ *we have* $x' \in {}^{co}\mathbf{I}$ *and* $d \in \mathbf{Sd}$ *such that* $x = \frac{d+x'}{2}$.
case differentiation by d*:*
If $d = 0$ *we are done, because then* $2x = x' \in {}^{co}\mathbf{I}$.
The cases $d = 1$ *and* $d = -1$ *are almost similar so we just consider* $d = -1$.
Here we have $2x = -1 + x'$ *and* $x' \geq 0$.

Therefore we show (CoIPosToCoIMinusOne):

$$\forall^{nc}_y . {}^{co}\mathbf{I} y \to y \geq 0 \to {}^{co}\mathbf{I}(y - 1)$$

Therefore we show (CoIPosToCoIMinusOne):

$$\forall^{nc}_{y}.{}^{co}\mathbf{I}y \to y \geq 0 \to {}^{co}\mathbf{I}(y-1)$$

Again we get $y' \in {}^{co}\mathbf{I}$ and $e \in \mathbf{Sd}$ such that $y = \frac{e+y'}{2}$ and we do case differentiation by e:
If $e = 1$ we get $y - 1 = \frac{1+y'}{2} - 1 = \frac{-1+y'}{2} \in {}^{co}\mathbf{I}$

Therefore we show (CoIPosToCoIMinusOne):

$$\forall_y^{nc}. {}^{co}\mathbf{I}y \to y \geq 0 \to {}^{co}\mathbf{I}(y-1)$$

Again we get $y' \in {}^{co}\mathbf{I}$ and $e \in \mathbf{Sd}$ such that $y = \frac{e+y'}{2}$ and we do case differentiation by e:
If $e = 1$ we get $y - 1 = \frac{1+y'}{2} - 1 = \frac{-1+y'}{2} \in {}^{co}\mathbf{I}$
If $e = 0$ we get $y - 1 = \frac{-1+(y'-1)}{2}$ and also $y' \leq 0$.
If $e = -1$ then $y' = 1$ must hold and then $y - 1 = -1 \in {}^{co}\mathbf{I}$. □

```
cCoIPosToCoIMinusOne u0 = strCoRec u0 f where
  f (SdR :~: u2) = (SdL, Left u2)
  f (SdM :~: u2) = (SdL, Right u2)
  f (SdL :~: u2) = (SdL, Left
  (strCoRec (RealConstr (const ( (-1) :#: One)) (const Zero))
   ((SdL , ) . Right)))
```

```
cCoIToCoIDouble :: Str -> Str
cCoIToCoIDouble u0 = let (s1 :~: u1) = u0 in case s1 of
  SdR -> cCoINegToCoIPlusOne u1
  SdM -> u1
  SdL -> cCoIPosToCoIMinusOne u1
```

Theorem

$$\forall^{nc}_{x,y}. {}^{co}\mathbf{I}x \to {}^{co}\mathbf{I}y \to \frac{1}{4} \leq y \to |x| \leq y \to {}^{co}\mathbf{I}\frac{x}{y}$$

Theorem

$$\forall^{nc}_{x,y}.{}^{co}\mathbf{I}x \to {}^{co}\mathbf{I}y \to \frac{1}{4} \le y \to |x| \le y \to {}^{co}\mathbf{I}\frac{x}{y}$$

Proof. *We distinguish the following cases:*

If $x = 1\tilde{x}$, $x = 01\tilde{x}$ *or* $x = 001\tilde{x}$ *we have* $x \ge 0$ *and* $\frac{x}{y} = \frac{1+\frac{x'}{y}}{2}$ *where* $x' := 4\frac{x-\frac{y}{2}}{2} = 2 \cdot 2\frac{x+0::(-y)}{2}$ *(ColDivSatColClAux1).*

If $x = \overline{1}\tilde{x}$, $x = 0\overline{1}\tilde{x}$ *or* $x = 00\overline{1}\tilde{x}$ *we have* $x \le 0$ *and* $\frac{x}{y} = \frac{-1+\frac{x'}{y}}{2}$ *where* $x' := 4\frac{x+\frac{y}{2}}{2} = 2 \cdot 2\frac{x+0::y}{2}$ *(ColDivSatColClAux4).*

If $x = 000\tilde{x}$ *we have* $|2x| \le 2\frac{1}{8} \le y$ *and* $\frac{x}{y} = \frac{0+\frac{x'}{y}}{2}$ *where* $x' = 2x$.

Theorem

$$\forall^{nc}_{x,y}.{}^{co}\mathbf{I}x \to {}^{co}\mathbf{I}y \to \frac{1}{4} \le y \to |x| \le y \to {}^{co}\mathbf{I}\frac{x}{y}$$

Proof. *We distinguish the following cases:*

If $x = 1\tilde{x}$, $x = 01\tilde{x}$ or $x = 001\tilde{x}$ we have $x \ge 0$ and $\frac{x}{y} = \frac{1+\frac{x'}{y}}{2}$ where $x' := 4\frac{x-\frac{y}{2}}{2} = 2 \cdot 2\frac{x+0::(-y)}{2}$ (ColDivSatColClAux1).

If $x = \overline{1}\tilde{x}$, $x = 0\overline{1}\tilde{x}$ or $x = 00\overline{1}\tilde{x}$ we have $x \le 0$ and $\frac{x}{y} = \frac{-1+\frac{x'}{y}}{2}$ where $x' := 4\frac{x+\frac{y}{2}}{2} = 2 \cdot 2\frac{x+0::y}{2}$ (ColDivSatColClAux4).

If $x = 000\tilde{x}$ we have $|2x| \le 2\frac{1}{8} \le y$ and $\frac{x}{y} = \frac{0+\frac{x'}{y}}{2}$ where $x' = 2x$.
In each case it is clear, that $|x'| \le y$ and so we just have to show ${}^{co}\mathbf{I}x'$. This follows from the last lemma and the last theorem. □

```
cCoIDiv u0 u1 = strCoRec u0 func where
  func u2@(d :~: e :~: f :~: _) =
    case d of
      SdR -> (SdR, Right (cCoIDivSatCoIClAux1 u2 u1))
      SdL -> (SdL, Right (cCoIDivSatCoIClAux4 u2 u1))
      SdM -> case e of
       SdR -> (SdR, Right (cCoIDivSatCoIClAux1 u2 u1))
       SdL -> (SdL, Right (cCoIDivSatCoIClAux4 u2 u1))
       SdM -> case f of
        SdR -> (SdR, Right (cCoIDivSatCoIClAux1 u2 u1))
        SdL -> (SdL, Right (cCoIDivSatCoIClAux4 u2 u1))
        SdM -> (SdM, Right (cCoIToCoIDouble u2))

cCoIDivSatCoIClAux1 u0 u1 = cCoIToCoIDouble $ cCoIToCoIDouble $
 cCoIAverage u0 $ cCoIUMinus $ strCoRec (SdM, u1) f where
  f (s, u) = (s, Left u)

cCoIDivSatCoIClAux4 u0 u1 = cCoIToCoIDouble $ cCoIToCoIDouble $
 cCoIAverage u0 $ strCoRec (SdM, u1) f where
  f (s, u) = (s, Left u)
```

Conclusion

We have constructed a SD representation of $\frac{x+y}{2}$ and $\frac{x}{y}$ out of the SD representation from x and y.

With the SD representation we can do exact real arithmetic simpler than with the binary representation.

To get finitly many digits of the output we just need finitly many digits of the input independet of the value of the imput.