Point-free Descriptive Set Theory and Algorithmic Randomness

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σ-frames . . .

A σ-frame $\mathcal{O}(X)$ is a partially-ordered set with:

- countable joins $\bigvee$ (including the empty join $\emptyset$),
- finite meets $\wedge$ (including the empty meet X),
- satisfying the countable distributive law:

$$U \wedge (\bigvee_i V_i) \;=\; \bigvee_i U \wedge V_i \;\;.$$

A morphism $p\colon \mathcal{O}(Y) \to \mathcal{O}(X)$, between σ-frames is a function that preserves countable joins and finite meets.

We write $\sigma\mathbf{Frm}$ for the category of σ-frames.

. . . and σ-locales

A σ-locale X is given by a σ-frame $\mathcal{O}(X)$.

A map $f\colon X \to Y$, between σ-locales X, Y, is given by a morphism $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ of σ-frames.

We write $\sigma\mathbf{Loc}$ for the category of σ-locales.

(N.B., $\sigma\mathbf{Loc} \simeq \sigma\mathbf{Frm}^{\mathsf{op}}$.)

Example σ-frames

- $\mathcal{O}(X)$ = the lattice of open subsets of a topological space.
- $\mathcal{O}(X)$ = the lattice of Borel subsets of a topological space.
- $\mathcal{O}(X)$ = the lattice of $\mathbf{\Sigma}_\alpha$-subsets of a topological space, for any ordinal α.

Full subcategories of σ**Loc**

σ**Loc** is the category of maps between σ-locales.

- The category of continuous functions between hereditarily Lindelöf sober topological spaces.

- The category of Borel-measurable functions between standard Borel spaces.

Complements and Boolean algebras

The complement (if it exists) of an element u in a distributive lattice P is the (necessarily unique) element $\overline{u} \in P$ satisfying

$$u \wedge \overline{u} = \bot \qquad\qquad u \vee \overline{u} = \top$$

If $p : P \to Q$ is a homomorphism of distributive lattices and $u, \overline{u}$ are complements in P then $p(u), p(\overline{u})$ are complements in Q.

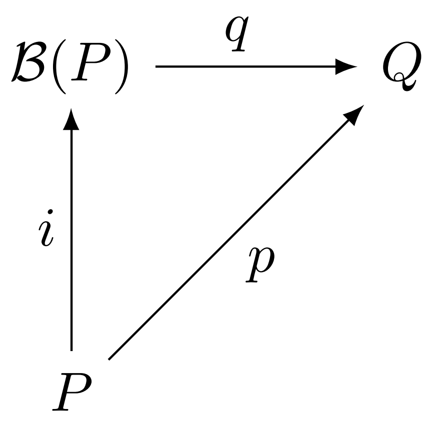
A distributive lattice P is a Boolean algebra if and only if every element of P has a complement.

Every distributive-lattice homomorphism $p : P \to Q$ between Boolean algebras is a Boolean-algebra homomorphism.

σ-Boolean algebras

For any σ-frame P, define a σ-Boolean algebra $\mathcal{B}(P)$ and homomorphism $i\colon P \longrightarrow \mathcal{B}(P)$ via the following universal property.

- for every homomorphism $p\colon P \to Q$, where Q is a σ-Boolean algebra, there exists a unique homomorphism q such that

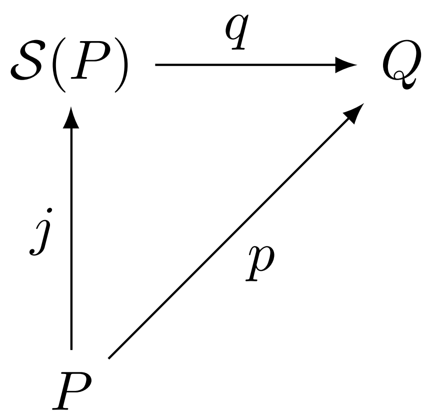


Theorem (CLASS). For any quasi-Polish space X, it holds that $\mathcal{B}(\mathcal{O}(X)) \cong \mathcal{B}or(X)$.

The ‘jump’ functor

For any σ-frame P, define a σ-frame $\mathcal{S}(P)$ and homomorphism $j\colon P \longrightarrow \mathcal{S}(P)$ via the following universal property.

1. Every element in the image of j has a complement in $\mathcal{S}(P)$; and
2. for every homomorphism $p\colon P \to Q$, where every element in the image of p has a complement in Q, there exists a unique homomorphism q such that



The Borel hierarchy

Theorem (CLASS). For any quasi-Polish space X, it holds that $\mathcal{S}^n(\mathcal{O}(X)) \cong \mathbf{\Sigma}_{n+1}(X)$.

- The classical result should generalise to ordinal-indexed iterations.
- It should hold constructively that $\mathcal{B}$ is the free monad over the functor $\mathcal{S}$.
- By interpreting the definition in suitable realizability toposes, it should be possible to obtain connections with Turing degrees, the arithmetic hierarchy and the lightface hierarchy.
- ...

Probability valuations

Write $\overrightarrow{[0,1]}$ for the set of ‘reals’ defined as limits of ascending sequences of rationals in $[0,1]$.

A probability (σ-)valuation on a σ-frame $\mathcal{O}(X)$ is a function

$$\mu : \mathcal{O}(X) \to \overrightarrow{[0,1]}$$

satisfying

- $\mu(\emptyset) = 0$ and $\mu(X) = 1$.
- $\mu(u \vee v) + \mu(u \wedge v) = \mu(u) + \mu(v)$.
- $u \leq v$ implies $\mu(u) \leq \mu(v)$.
- $(u_i)_i$ ascending implies $\mu(\bigvee_i u_i) = \sup_i \mu(u_i)$.

The Cantor locale

Define $\mathcal{O}(\mathbf{2}^{\mathbf{N}})$ to be the free σ-frame on countably many complemented generators $(c_i)_i$.

Intuitively, the generator c_i represents the clopen set

$$\{\alpha \in \{0,1\}^\omega \mid \alpha_i = 1\}$$

Proposition. There is a unique probability valuation $\lambda \colon \mathcal{O}(\mathbf{2}^{\mathbf{N}}) \to \overrightarrow{[0,1]}$ such that $\lambda(c_i) = \frac{1}{2}$ for every i.

We are endowing the Cantor (σ-)locale $\mathbf{2}^{\mathbf{N}}$ with the uniform probability valuation.

σ-sublocales

A map $f\colon X \to Y$ between σ-locales is said to be an embedding if $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective.

The embeddings determine the notion of σ-sublocale.

The σ-sublocales of a σ-locale X are in 1–1 correspondence with congruences on $\mathcal{O}(X)$.

The σ-sublocales of X form a complete lattice $\mathcal{S}ub(X)$ under the embedding order.

Open σ-sublocales

For $v \in \mathcal{O}(X)$ define a congruence relation $\approx_{o(v)}$ on $\mathcal{O}(X)$ by

$$u \approx_{o(v)} u' \;\Leftrightarrow\; u \wedge v = u' \wedge v$$

It holds that

$$\mathcal{O}(X)/\approx_{o(v)} \;\cong\; \downarrow v$$

We call $o(v)$ the open σ-sublocale determined by v.

The σ-locale of random sequences

For $u, v \in \mathcal{O}(\mathbf{2}^{\mathbf{N}})$, define:

$$u \approx v \;\Leftrightarrow\; \lambda(u) = \lambda(u \wedge v) = \lambda(v)$$

Define $\mathcal{O}(\mathrm{Ran}) \;=\; \mathcal{O}(\mathbf{2}^{\mathbf{N}})/\approx$.

Theorem

1. Ran is the intersection of all outer-measure-1 σ-sublocales of $\mathbf{2}^{\mathbf{N}}$.
2. Ran is the intersection of all measure-1 open σ-sublocales of $\mathbf{2}^{\mathbf{N}}$.
3. Ran itself has outer measure 1.

(We say a σ-sublocale $X \subseteq \mathbf{2}^{\mathbf{N}}$ has outer measure 1 if, for every open σ-sublocale $X \subseteq o(u) \subseteq \mathbf{2}^{\mathbf{N}}$, it holds that $\lambda(u) = 1$.)

Points

The terminal σ-locale $\mathbf{1}$ is given by defining $\mathcal{O}(\mathbf{1})$ to be the free σ-frame on no generators.

A point of a σ-locale X is a map from the terminal σ-locale $\mathbf{1}$ to X.

That is, points are given by σ-frame homomorphisms from $\mathcal{O}(X)$ to $\mathcal{O}(\mathbf{1})$.

Proposition Ran has no points.

Classical points

In our intuitionistic development, there is a potentially weaker notion of classical point of a σ-locale X: a map from $\mathbf{1}^c$ to X where

$$\mathcal{O}(\mathbf{1}^c) \;=\; \Omega_{\neg\neg} \;=\; \{p \in \Omega \mid \neg\neg p \Rightarrow p\}$$

Under classical logic, $\mathbf{1}^c \cong \mathbf{1}$, so classical points coincide with points.

If LEM fails, they may differ.

We can view this difference by interpreting the development in Hyland's effective topos $\mathcal{E}\!f\!f$. [Hyland 1981]

Interpretation in $\mathcal{E}\!\mathit{ff}$

The objects in our development all produce assemblies.

$$
\begin{aligned}
|\mathcal{O}(\mathbf{2}^{\mathbf{N}})| &= \{U \subseteq \{0,1\}^{\omega} \mid U \text{ c.e. open}\} \\
n \,\mathbf{r}\, U &\Leftrightarrow n \text{ codes a sequence } (C_i)_i \text{ of cylinders s.t. } U = \bigcup_i C_i \\
|\overrightarrow{[0,1]}| &= \{x \in [0,1] \mid x \text{ left c.e.}\} \\
n \,\mathbf{r}\, x &\Leftrightarrow n \text{ codes a sequence } (q_i)_i \text{ of rationals s.t. } x = \sup_i q_i \\
|\mathcal{O}(\mathbf{1})| &= \{0,1\} \\
n \,\mathbf{r}\, a &\Leftrightarrow (a = 1 \wedge n \in K) \vee (a = 0 \wedge n \in \overline{K}) \\
|\mathcal{O}(\mathbf{1}^c)| &= \{0,1\} \\
n \,\mathbf{r}\, a &\Leftrightarrow \text{true}
\end{aligned}
$$

Theorem (CLASS)

1. The points of $\mathbf{2}^{\mathbf{N}}$ in $\mathcal{E}\!f\!f$ are in 1–1 correspondence with computable sequences $\alpha \in \{0,1\}^\omega$.
2. The classical points of $\mathbf{2}^{\mathbf{N}}$ in $\mathcal{E}\!f\!f$ are in 1–1 correspondence with arbitrary sequences $\alpha \in \{0,1\}^\omega$.
3. Ran in $\mathcal{E}\!f\!f$ has no points.
4. The classical points of Ran in $\mathcal{E}\!f\!f$ are in 1–1 correspondence with Kurtz random sequences $\alpha \in \{0,1\}^\omega$.

A sequence $\alpha \in \{0,1\}^\omega$ is Kurtz random if it is contained in every measure-1 c.e. open subset $U \subseteq \{0,1\}^\omega$. [Kurtz 1981]

Revisiting constructive point-free descriptive set theory

There are two possible approaches to generating a σ-frame $\Sigma_{\alpha+1}(X)$ from $\Sigma_\alpha(X)$.

1. Obtain $\Sigma_{\alpha+1}(X)$ as the free σ-frame that adds complements to every element of $\Sigma_\alpha(X)$.

   This is the 'jump' operation from earlier.

2. Obtain $\Sigma_{\alpha+1}(X)$ by extending the σ-coframe $\Pi_\alpha(X) = (\Sigma_\alpha(X))^{\mathrm{op}}$ with countable joins.

Approach 1 seems the 'correct' approach to obtaining a rich constructive point-free descriptive set theory.

But we now follow approach 2.

The σ-frame $\Sigma_2(\mathbf{2}^{\mathbf{N}})$

Define $\Sigma_2(\mathbf{2}^{\mathbf{N}})$ to be the free σ-frame over $\mathcal{O}(\mathbf{2}^{\mathbf{N}})^{\mathsf{op}}$ considered as a distributive lattice.

- There is a distributive-lattice homomorphism

$$c\colon \mathcal{O}(\mathbf{2}^{\mathbf{N}})^{\mathsf{op}} \to \Sigma_2(\mathbf{2}^{\mathbf{N}})$$

- It further holds that c preserves countable meets.

We call elements of $\Sigma_2(\mathbf{2}^{\mathbf{N}})$ in the image of c closed.

The Σ_2-reals

Define $\overleftrightarrow{[0,1]}$ to be the set of 'reals' obtained as nested sup-infs of doubly indexed sequences of rationals in $[0,1]$.

Proposition. The probability valuation

$$\lambda\colon \mathcal{O}(\mathbf{2^N})^{\mathsf{op}} \to \overleftarrow{[0,1]}$$

extends to a 'probability valuation'

$$\lambda\colon \Sigma_2(\mathbf{2^N}) \to \overleftrightarrow{[0,1]}$$

Moreover, λ preserves countable meets of closed elements.

Two random sub-Σ_2-locales

Let Ran_1 be the intersection of all measure-1 Σ_2 sub-Σ_2-locales of $\Sigma_2(\mathbf{2^N})$.

Let Ran_2 be the intersection of all outer-measure-1 sub-Σ_2-locales of $\Sigma_2(\mathbf{2^N})$.

It is immediate that $\mathrm{Ran}_2 \subseteq \mathrm{Ran}_1 \subseteq \mathbf{2^N}$.

Proposition. Ran_2 (hence Ran_1) has outer measure 1.

Theorem (CLASS).

1. The classical Σ_2-points of Ran_1 in $\mathcal{E}\!f\!f$ are in 1–1 correspondence with Martin-Löf random sequences $\alpha \in \{0,1\}^\omega$.

2. The classical Σ_2-points of Ran_2 in $\mathcal{E}\!f\!f$ are in 1–1 correspondence with difference random sequences $\alpha \in \{0,1\}^\omega$.

A sequence α is not ML-random if and only if, for every confidence level $\epsilon > 0$, there exists (computably in ϵ) a c.e. open U_ϵ with measure $< \epsilon$ such that $\alpha \in U_\epsilon$. [Martin-Löf 1966]

A sequence α is difference random if and only if it is ML-random and the halting set K is not computable relative to α. [Franklin & Ng 2011]

Some further directions

- Canonicity theorems
- Point-free and constructive measure extension theorems
- Formally develop the parallel between increasing complexity of sets and increasing complexity of real numbers