Sequentially locally convex QCB-spaces and Complexity Theory

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Contents

Sequentially locally convex QCB-spaces

Remember

- *Topological vector space*: a vector space endowed with a topology rendering addition & scalar multiplication continuous.
- *Locally convex space*: a topological vector space whose topology is induced by seminorms.

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- *Topological vector space*: a vector space endowed with a topology rendering addition & scalar multiplication continuous.
- *Locally convex space*: a topological vector space whose topology is induced by seminorms.
- *Seminorm* on $\mathfrak{X}$: a function $p\colon \mathfrak{X} \to \mathbb{R}_{\geq 0}$ s.t.
  - $p(\vec{0}) = 0$,
  - $p(x + y) \leq p(x) + p(y)$,
  - $p(\alpha \cdot x) = |\alpha| \cdot p(x)$.
- p is a *norm*, if additionally $p(x) = 0 \implies x = \vec{0}$.

Example (Locally convex spaces)

- Any normed space.
- The space $\mathcal{D}$ of test functions on $\mathbb{R}$.

Remember

- QCB-spaces = the class of topological spaces which can be handled by TTE, the Type Two Model of Effectivity.
- QCB-space: a **q**uotient of a **c**ountably **b**ased top. space.

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Facts

- Separable metrisable spaces are QCB-spaces.
- The quotient topology of a TTE-representation is *QCB*.
- The category QCB of QCB-spaces and continuous functions has excellent closure properties:
  - cartesian closed
  - countably complete
  - countably co-complete

Why not just locally convex QCB-spaces?

Why not just locally convex QCB-spaces?

Problem

- Important locally convex spaces are not sequential.
- Locally convex QCB-spaces do not enjoy nice closure properties.

Example

The vector space $\mathcal{D}$ of test functions on $\mathbb{R}$.

- The standard locally convex topology on $\mathcal{D}$ is not sequential, hence not *QCB*.
- Its sequentialisation is *QCB*, but not locally convex.

Definition

A *sequentially locally convex QCB-space* $\mathfrak{X}$ is

- a vector space
- endowed with a QCB_0-topology
- such that the convergence relation is induced by a family of continuous seminorms.

Abbreviation: *QLC*-space.

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Abbreviation: *QLC*-space.

Remark

- Any QLC-space is the sequentialisation of a locally convex space.
- Sequentialisation $seq(\tau)$ of a topology τ:
  the family of all *sequentially open* sets pertaining to τ.

Proposition

Let $\mathfrak{X}$ be a sequentially locally convex QCB-space. Then:

- $\mathfrak{X}$ is Hausdorff.
- Scalar multiplication is topologically continuous.
- Vector addition is *sequentially continuous*,
- but not necessarily topologically continuous.

Remember

$f\colon \mathsf{X} \to \mathsf{Y}$ is *sequentially continuous*, if $(x_n)_n \to x_\infty$ in X implies $\big(f(x_n)\big)_n \to f(x_\infty)$ in Y.

Example (QLC-spaces)

- separable Banach spaces
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- The sequentialisation of the standard locally convex topology τ_{LC} on $\mathcal{D}$ is *QCB*.
- Hence $\mathcal{D}$ endowed with $seq(\tau_{\mathrm{LC}})$ is a QLC-space.

Example (QLC-spaces)

- separable Banach spaces
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Example

Let $\mathcal{D}$ be the vector space of test functions on $\mathbb{R}$.

- The sequentialisation of the standard locally convex topology τ_{LC} on $\mathcal{D}$ is *QCB*.
- Hence $\mathcal{D}$ endowed with $seq(\tau_{\mathrm{LC}})$ is a QLC-space.
- Vector addition is not topologically continuous w.r.t. $seq(\tau_{\mathrm{LC}})$,
- but sequentially continuous.
- $seq(\tau_{\mathrm{LC}})$ is not locally convex.

Definition

Denote by QLC the following category:

- *Objects:*
  all sequentially locally convex QCB-spaces
- *Morphisms:*
  all continuous & linear functions $f\colon \mathfrak{X} \to \mathfrak{Y}$

Theorem

The category QLC is cartesian and monoidal closed:

- cartesian product $\mathfrak{X} \times \mathfrak{Y}$
- function space $\mathfrak{Lin}(\mathfrak{X}, \mathfrak{Y})$
- tensor product $\mathfrak{X} \otimes \mathfrak{Y}$

Proof Sketch

Use the corresponding constructions in QCB.

Topological dual

- *Topological dual* $\mathfrak{X}'$ of a topological vector space $\mathfrak{X}$:

$$\{ f\colon \mathfrak{X} \to \mathbb{R} \mid f \text{ continuous \& linear} \}$$

- There are several ways to topologise $\mathfrak{X}'$.

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- There are several ways to topologise $\mathfrak{X}'$.

The dual space $\mathfrak{X}^*$ in QLC

- *Underlying vector space of* $\mathfrak{X}^*$:
  $\{f : \mathfrak{X} \to \mathbb{R} \mid f \text{ continuous \& linear}\}$
- *Topology of* $\mathfrak{X}^*$:
  The subspace topology of the QCB-function space $\mathbb{R}^{\mathfrak{X}}$

Duals in QLC

Proposition

- If $\mathfrak{X}$ is finite-dimensional, then $\mathfrak{X}^* \cong \mathfrak{X}$.
- If $\mathfrak{X}$ is a separable Banach space, then
  - $\mathfrak{X}^*$ need not be the Banach space dual,
  - $\mathfrak{X}^*$ carries the sequentialisation of the weak-$*$-topology.
- If $\mathfrak{X}$ is separable normed, then $\mathfrak{X}^{**}$ is the completion of $\mathfrak{X}$.

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Proposition

If $\mathfrak{X} \in$ QLC is metrisable, then $\mathfrak{X}^*$ is co-Polish.

Co-Polish spaces

Definition

We call a QCB-space X *co-Polish*, if $\mathbb{S}^X$ is quasi-Polish.

Remark

- quasi-Polish $=$ separable completely quasi-metrisable
- $\mathbb{S}$ denotes the Sierpiński space

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Theorem (Characterisation)

Let X be a Hausdorff QCB-space. TFAE:

- X is co-Polish.
- $\mathbb{S}^X$ is has a countable base.
- X has an admissible TTE-representation with a locally compact domain.
- X is the direct limit of an increasing sequence of compact metrisable spaces.

Proposition

Let X be a Hausdorff space with a countable base. Then:

- X is co-Polish $\iff$ X is locally compact.

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Proposition

- The category of co-Polish Hausdorff spaces
  - has finite products and equalisers (inherited from QCB),
  - but is not closed under forming QCB-exponentials.
- Hausdorff quotients of co-Polish Hausdorff spaces are co-Polish.

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- Hausdorff quotients of co-Polish Hausdorff spaces are co-Polish.
- For any Y with a countable base and any co-Polish space X, Y^X has a countable base.
- [de Brecht & Sch.] For any (quasi-)Polish space Y and any co-Polish space X, Y^X is (quasi-)Polish.

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- [de Brecht & Sch.] For any (quasi-)Polish space Y and any co-Polish space X, Y^X is (quasi-)Polish.
- A topological subspace Y of a co-Polish Hausdorff space X is co-Polish iff Y is a crescent subset of X.

Co-Polish spaces in QLC

Theorem

Let $\mathfrak{X}$ be a sequentially locally convex QCB-space. Then:

- $\mathfrak{X}$ is sep. metrisable $\Longleftrightarrow$ $\mathfrak{X}^*$ is co-Polish
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  $\Longleftrightarrow$ $\mathfrak{X}^*$ is Polish

Proposition

Any co-Polish QLC-space is locally convex.

Application in Type 2 Complexity Theory

Type Two Model of Effectivity (TTE)

- A *representation* of X is a partial surjection $\delta\colon \Sigma^{\mathbb{N}} \dashrightarrow X$.

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- A *representation* of X is a partial surjection $\delta\colon \Sigma^{\mathbb{N}} \dashrightarrow X$.
- Let $\delta\colon \Sigma^{\mathbb{N}} \dashrightarrow X$ and $\gamma\colon \Sigma^{\mathbb{N}} \dashrightarrow Y$ be representations.

  $f\colon X \to Y$ is called *(δ, γ)-computable*, if there is a computable function $g\colon \Sigma^{\mathbb{N}} \dashrightarrow \Sigma^{\mathbb{N}}$ such that

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ \delta \big\uparrow & \circlearrowleft & \big\uparrow \gamma \\ \Sigma^{\mathbb{N}} & \xrightarrow[\;g\;]{} & \Sigma^{\mathbb{N}} \end{array}$$

  commutes.
- $g\colon \Sigma^{\mathbb{N}} \dashrightarrow \Sigma^{\mathbb{N}}$ is *computable*, if there is an oracle Turing machine M that computes g.

Definition

Let M be an oracle machine that (δ, γ)-computes $f\colon X \to Y$.

- Define the *computation time* of M by

$$Time_M(p, n) := \left\{ \begin{array}{l} \text{the number of steps that } M \text{ makes} \\ \text{on input } (p, n) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \end{array} \right.$$

- Define the *relative computation time* on $A \subseteq X$ by

$$Time^{\delta}_M(A, n) := \sup \left\{ Time_M(p, n) \,\middle|\, \delta(p) \in A \right\}$$

Definition

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Problem

- The sup may be equal to ∞, even if $A = \{x\}$.
- So M may not even have a time bound on singletons.

How to ensure the existence of time bounds?

Observation

- If $\delta^{-1}[A]$ is compact, then $Time^{\delta}_{M}(A, n) < \infty$.
- If δ is a continuous representation of a space X, then $\delta^{-1}[A]$ compact $\implies$ A compact.

How to ensure the existence of time bounds?

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Definition

A continuous representation δ of X is called *proper*, if $\delta^{-1}[K]$ is compact for every compact $K \subseteq \mathsf{X}$.

Lemma

For a proper δ, time complexity can be measured by a function

$$T\colon \{K \subseteq \mathsf{X} \mid K \text{ compact}\} \times \mathbb{N} \to \mathbb{N}$$

Example

The signed-digit representation ϱ_{sd} for $\mathbb{R}$ defined by

$$\varrho_{\mathrm{sd}}(p) := p(0) + \sum_{i=1}^{\infty} p(i) \cdot 2^{-i} \qquad \text{for } p \in \mathbb{Z} \times \{-1, 0, 1\}^{\mathbb{N}}$$

is proper.

Theorem

A sequential space X has a proper admissible representation iff X is separable metrisable.

Simple Complexity

Aim

Measurement of time complexity in terms of

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Idea

- Equip δ with a "size function" $S\colon \mathrm{dom}(\delta) \to \mathbb{N}$.
- Measure time complexity by $T_M\colon \mathbb{N} \times \mathbb{N} \dashrightarrow \mathbb{N}$ defined by

$$T_M(a, n) := \sup \{ Time_M(p, n) \mid S(p) = a \},$$

where M is a realising machine.

Definition

We call $S\colon \mathrm{dom}(\delta) \to \mathbb{N}$ a *size function* for δ, if

- S is continuous,
- $S^{-1}\{a\}$ is compact for all $a \in \mathbb{N}$.

Example

Natural size functions for the signed-digit representation for $\mathbb{R}$:

- $S_1(p) = |p(0)|$
- $S_2(p) = \log_2(|p(0)| + 1)$

Lemma

Let δ be a representation with size function S. Then

$$T_M(a,n) = \sup \left\{ Time_M(p,n) \,\middle|\, S(p) = a \right\}$$

exists for all $a, n \in \mathbb{N}$, whenever M realises a total function on X.

Corollary

Time complexity of a function f on (X, δ) can be measured in two *discrete* parameters:

- the size $S(p)$ of the input name p &
- the desired output precision.

Example

Let $\mathcal{P}$ be the vector space of polynomials over the reals.

- Suitable representation $\varrho_{\mathcal{P}}$:
  - Store the coefficients & an upper bound of the degree
- Size $S(q) \in \mathbb{N} \times \mathbb{N}$ of a name q:
  - the upper bound of the degree & the maximum of the integer parts of the coefficients

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  in time polynomial in the size functions of $\varrho_{\mathcal{P}}$ and ϱ_{sd}.

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- Evaluation is $(\varrho_{\mathcal{P}}, \varrho_{\mathrm{sd}}, \varrho_{\mathrm{sd}})$-computable in time polynomial in the size functions of $\varrho_{\mathcal{P}}$ and ϱ_{sd}.
- The final topology of $\varrho_{\mathcal{P}}$ is co-Polish, but neither metrisable nor countably-based.

Lemma
A representation has a size function iff its domain is locally compact.

Theorem
A Hausdorff QCB-space has an admissible representation with a size function iff it is co-Polish.

Application

- Let $\mathcal{E}$ be the space of infinitely differentiable functions on $\mathbb{R}$.
- $\mathcal{E}$ is a separable Fréchet space.

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- $\mathcal{E}$ is a separable Fréchet space.
- Hence $\mathcal{E}^*$ is a locally convex co-Polish space.
- $\mathcal{E}^*$ admits a simple measurement of complexity.

Application

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- $\mathcal{E}$ is a separable Fréchet space.
- Hence $\mathcal{E}^*$ is a locally convex co-Polish space.
- $\mathcal{E}^*$ admits a simple measurement of complexity.

Remark

- $\mathcal{E}^*$ can be identified with the space of distributions over $\mathbb{R}$ with compact support.
- The space $\mathcal{S}^*$ of tempered distributions is co-Polish, because $\mathcal{S}$ is a separable Fréchet space.
- The space $\mathcal{D}^*$ of all distributions is not co-Polish, as the space $\mathcal{D}$ of test functions is not countably-based.

Hybrid Representations

Observation

Representations for spaces in Functional Analysis are typically constructed by encoding:

- a sequence of reals &
- a sequence of discrete information.

Definition

- Let $\mathbb{H} := [-1;1]^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$.
- A *hybrid representation* of X is a partial surjection $\psi\colon \mathbb{H} \dashrightarrow X$.

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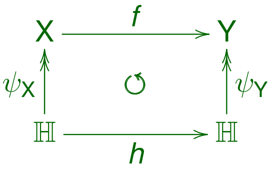
- Let $\mathbb{H} := [-1;1]^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$.
- A *hybrid representation* of X is a partial surjection $\psi\colon \mathbb{H} \dashrightarrow \mathsf{X}$.
- $f\colon \mathsf{X} \to \mathsf{Y}$ is $(\psi_{\mathsf{X}}, \psi_{\mathsf{Y}})$-computable, if there is a computable $h\colon \mathbb{H} \dashrightarrow \mathbb{H}$ such that



Example

- $\mathcal{C}[0;1]$:
  Choose a dense sequence $(d_i)_i$ in $[0;1]$. Define

$$\psi(r,p) = f \;:\Longleftrightarrow\; \begin{cases} \forall i \in \mathbb{N}.\, r(i) \cdot p(0) = f(d_i) \quad \& \\ k \mapsto p(k+1) \text{ is a modulus} \\ \qquad\qquad\qquad \text{of continuity for } f \end{cases}$$

- The space of polynomials $\mathcal{P}$:
  Use $\mathbb{H}_0 = [-1;1]^{\mathbb{N}} \times \mathbb{N}$ and define

$$\psi(r, \langle a, b\rangle) = P \;:\Longleftrightarrow\; P(\mathrm{x}) = \sum_{k=0}^{b} a \cdot r(k) \cdot \mathrm{x}^k$$

Definition

Let M be an oracle machine realising $f\colon (\mathsf{X}, \psi_{\mathsf{X}}) \to (\mathsf{Y}, \psi_{\mathsf{Y}})$.

A function $t\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 \to \mathbb{N}$ is a *time bound* for M, if

- for all $(r, p) \in \mathrm{dom}(\psi_{\mathsf{X}})$ and all $j, k \in \mathbb{N}$
- M produces $q(j)$ and some 2^{-k}-approximation to $s(j)$
  (where (s, q) denotes the produced representative of the result)
- in $\leq t(p, j, k)$ steps.

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Remark

Hybrid representations have an implicit size function

$$S\colon \mathbb{H} \to \mathbb{N}^{\mathbb{N}}, \ (r, p) \mapsto p.$$

Theorem

Any oracle Turing machine realising some function w.r.t. hybrid representations with closed domain has a continuous time bound $t\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 \to \mathbb{N}$.

The proof is based on:

Lemma

A hybrid representation ψ has a closed domain iff

$$\{(r,p) \in \mathrm{dom}(\psi) \mid p \in K\} \text{ is compact}$$

for every compact $K \subseteq \mathbb{N}^{\mathbb{N}}$.

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for every compact $K \subseteq \mathbb{N}^{\mathbb{N}}$.

Definition

A hybrid representation is *complete*, if its domain is closed.

Theorem

- A metric space has an admissible complete hybrid representation iff it is Polish.
- A Hausdorff space has an admissible complete hybrid representation over $\mathbb{H}_0 = [-1;1]^{\mathbb{N}} \times \mathbb{N}$ iff it is co-Polish.

Theorem

The category of Hausdorff QCB-spaces having an admissible complete hybrid representation has

- countable products,
- countable co-products,
- equalisers.

But it is not closed under forming function spaces in QCB.

Summary

- QLC-spaces provide a nice framework to study computability on locally convex spaces.
- Co-Polish Hausdorff spaces allow the measurement of complexity by natural number functions.
- Important examples are the duals of separable metrisable locally convex spaces.
- Hybrid representations yield a unifying approach to Complexity Theory in Computable Analysis.