Nonstandard Analysis, Computability Theory, and metastability

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Metastability trade-off: introducing a finite domain yields uniform and computable results.

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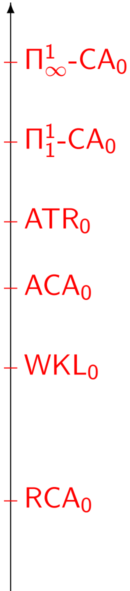

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NO type two functional computes Θ (Same for ζ as in $\mathsf{MCT}_{\mathsf{meta}}(\zeta)$)

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There is $\Psi^{2\rightarrow 1^*}$ such that for $f : [0,1] \rightarrow \mathbb{R}$ with modulus of continuity H^2 and $f(0)f(1) < 0$, $(\forall G^2)(\exists x \in \Psi(G,H))\big(x \in I \wedge |f(x)| < \frac{1}{G(x)}\big)$.

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There is Ψ such that for $f : [0,1] \to \mathbb{R}$ with modulus of uniform continuity h, and $S_n(f) := \sum_{i=0}^{2^n} f(\frac{i}{2^n})\frac{1}{2^n}$, we have

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Normann-Sanders I

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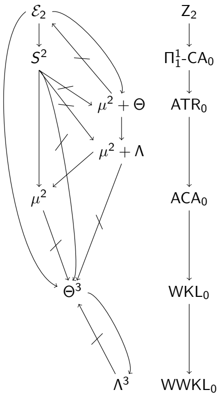

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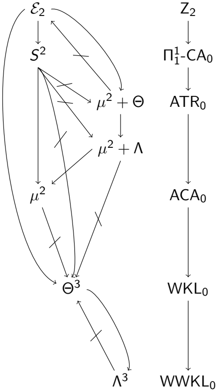

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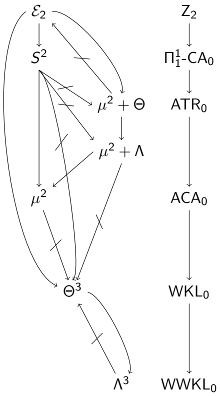

Λ is the functional from ‘metastable WWKL’, where the latter is weak weak König’s lemma. Where do Θ and Λ come from?

Nonstandard Analysis: it all started with . . .

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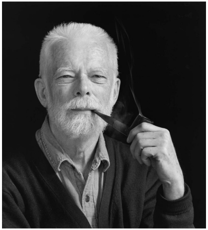

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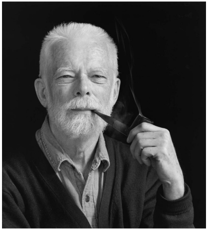

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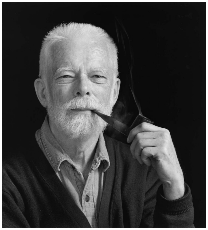

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Nelson: Internal Set Theory (1977): well-known axiomatic approach to Nonstandard Analysis based on ZFC set theory.

Introducing Nonstandard Analysis

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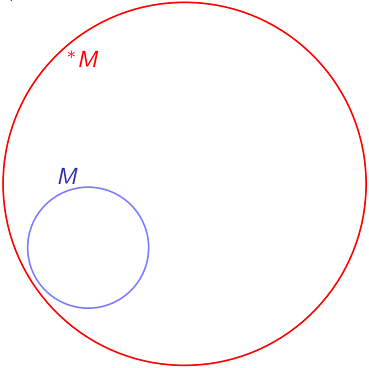

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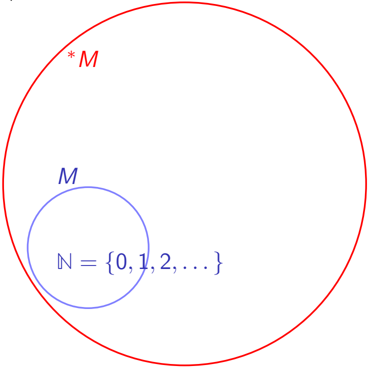

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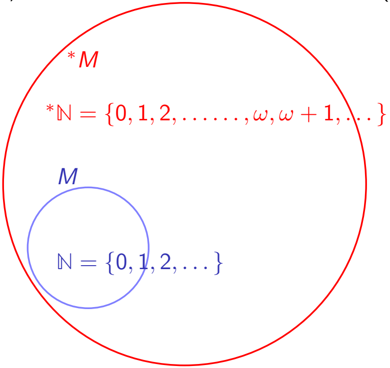

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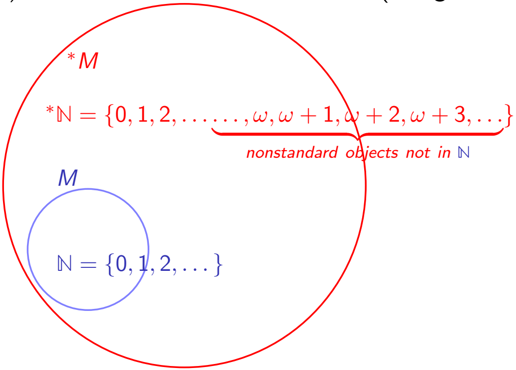

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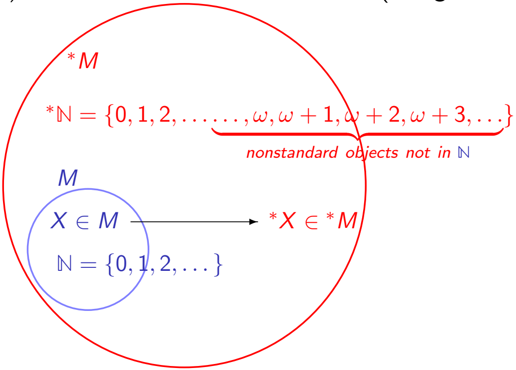

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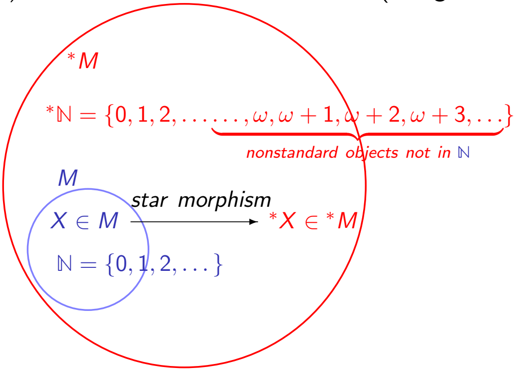

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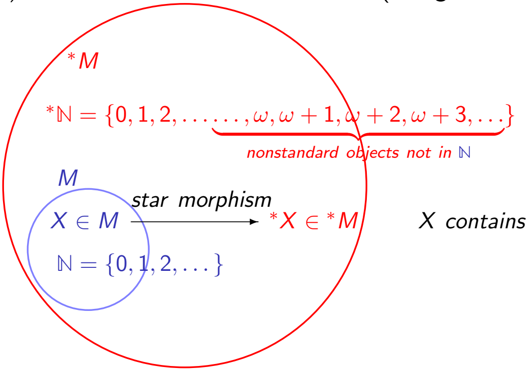

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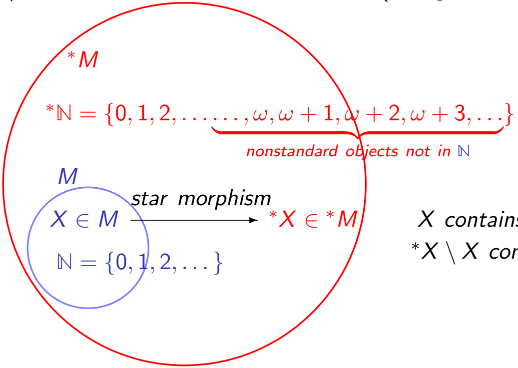

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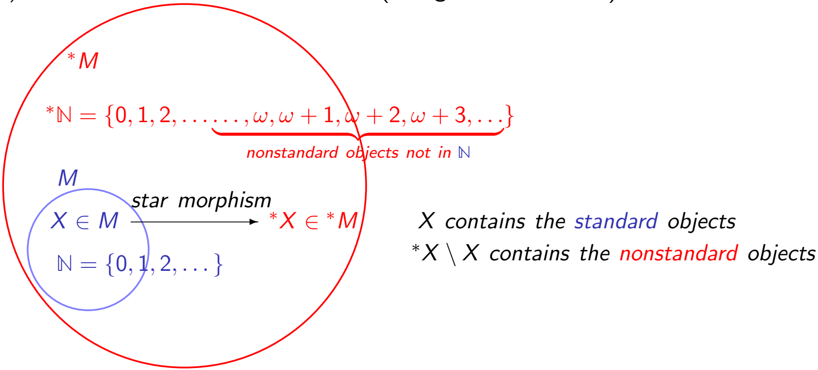

Three important properties connecting M and ${}^*M$:

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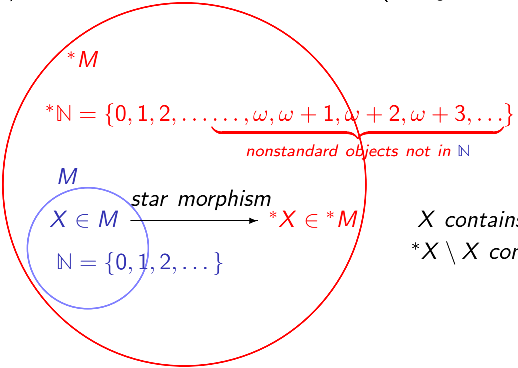

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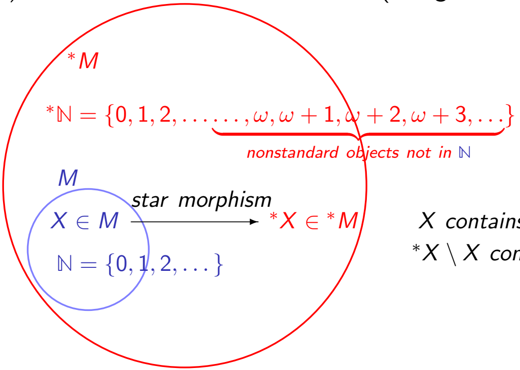

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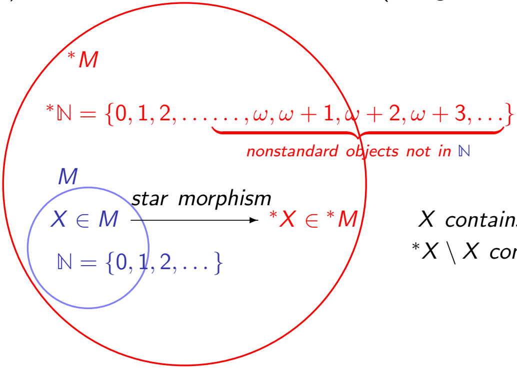

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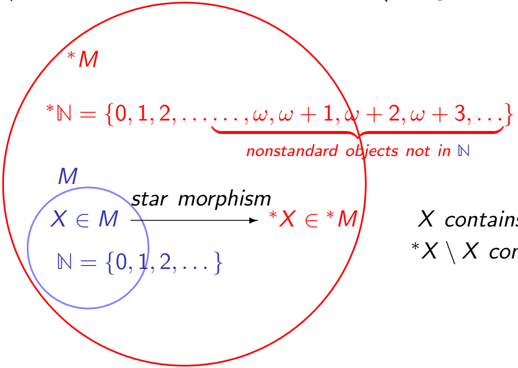

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3) Idealization/Saturation ...

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Over ZFC + Idealization, Transfer $\not\rightarrow$ Standard Part.

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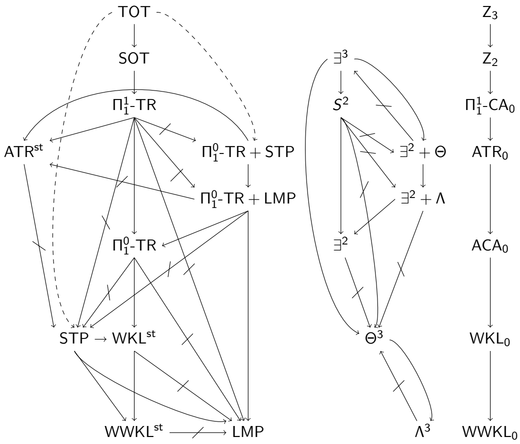
The functionals μ^2 and S^2 are realisers for Π^0_1-TR and Π^1_1-TR.

Normann-Sanders I

Results in `https://arxiv.org/abs/1702.06556`:



The functional interpretation IST to ZFC translates (most of) the left to the right, and vice versa (via the standardness of computations).

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And credit where it is due: OMKN

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ADVERTISING: Most of classical NSA has computational content (like constructive math); for a gentle introduction, see my arXiv paper:

To be or not to be constructive,
that is not the question.

to appear in the 2017 Brouwer volume and Indagationes Mathematicae.