Open determinacy and the perfect tree theorem

Takayuki Kihara & **Arno Pauly**

Nagoya University
Université Libre de Bruxelles

CCC 2017

Outline

The theorems

The context

The results

Outline

The theorems

The context

The results

Open determinacy

Definition

A Σ^0_1-game in $\mathbb{N}^{\mathbb{N}}$ is given by a winning condition $W \subseteq \mathbb{N}^*$. Two players take turns playing natural numbers. If the finite word w of numbers played so far ever falls into W, Player 1 wins. If this never happens, Player 2 wins.

Theorem

In every Σ^0_1-game, one of the players has a winning strategy.

Open determinacy

Definition

A Σ^0_1-game in $\mathbb{N}^\mathbb{N}$ is given by a winning condition $W \subseteq \mathbb{N}^*$. Two players take turns playing natural numbers. If the finite word w of numbers played so far ever falls into W, Player 1 wins. If this never happens, Player 2 wins.

Theorem

In every Σ^0_1-game, one of the players has a winning strategy.

Perfect Tree Theorem

Definition

A tree $T \subseteq \mathbb{N}^*$ is perfect, if it is non-empty and for any $v \in T$ there are incomparable extensions $v_1, v_2 \in T$.

Theorem (Perfect Tree Theorem)

For any tree T, either $[T]$ is countable or T has a perfect subtree.

Perfect Tree Theorem

Definition

A tree $T \subseteq \mathbb{N}^*$ is perfect, if it is non-empty and for any $v \in T$ there are incomparable extensions $v_1, v_2 \in T$.

Theorem (Perfect Tree Theorem)

For any tree T, either $[T]$ is countable or T has a perfect subtree.

Outline

The theorems

The context

The results

The idea behind Weihrauch reducibility

1. Identify a theorem

$$\forall x \in \mathbf{X}\ \exists y \in \mathbf{Y}\ .\ D(x) \Rightarrow\ T(x, y)$$

   with the multi-valued function $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $\mathrm{dom}(T) = D$ obtained by Skolemization.

2. Then compare theorems via Weihrauch-reducibility to learn about their *constructive content*.

The idea behind Weihrauch reducibility

1. Identify a theorem

$$\forall x \in \mathbf{X}\ \exists y \in \mathbf{Y}\ .\ D(x) \Rightarrow\ T(x,y)$$

   with the multi-valued function $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $\mathrm{dom}(T) = D$ obtained by Skolemization.

2. Then compare theorems via Weihrauch-reducibility to learn about their *constructive content*.

The idea behind Weihrauch reducibility

1. Identify a theorem

   $$\forall x \in \mathbf{X}\ \exists y \in \mathbf{Y}\ .\ D(x) \Rightarrow\ T(x, y)$$

   with the multi-valued function $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $\mathrm{dom}(T) = D$ obtained by Skolemization.
2. Then compare theorems via Weihrauch-reducibility to learn about their *constructive content*.

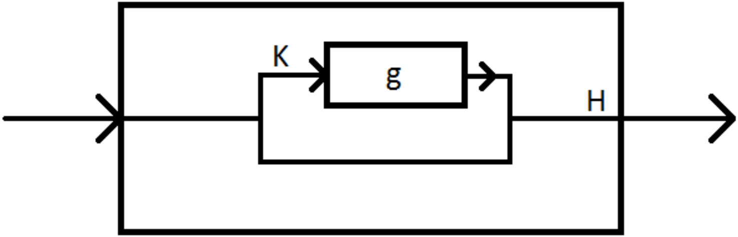
Weihrauch-reducibility

Definition

For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{V} \rightrightarrows \mathbf{W}$ say

$$f \leq_W g$$

iff there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $H\langle \mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, GK \rangle$ is a realizer of f for every realizer G of g.



The idea behind reverse mathematics

1. Fix some weak axiom system (RCA_0).
2. For theorems of second-order arithmetic, find canonic representatives such that RCA_0 proves their equivalence.
3. Big Five: RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1{-}\mathrm{CA}$.

The idea behind reverse mathematics

1. Fix some weak axiom system (RCA_0).
2. For theorems of second-order arithmetic, find canonic representatives such that RCA_0 proves their equivalence.
3. Big Five: RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1{-}\mathrm{CA}$.

The idea behind reverse mathematics

1. Fix some weak axiom system (RCA_0).
2. For theorems of second-order arithmetic, find canonic representatives such that RCA_0 proves their equivalence.
3. Big Five: RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1-\mathrm{CA}$.

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Comparison

Weihrauch reducibility:

- resource sensitive,
- varies under e.g. contraposition,
- absolute

Reverse mathematics:

- not resource sensitive,
- invariant under logical operations,
- not absolute

Operations on Weihrauch degrees

Let $f : \mathbf{X} \rightrightarrows \mathbf{Y}$, $g : \mathbf{U} \rightrightarrows \mathbf{Y}$

- $\widehat{f} : \mathbf{X}^\omega \rightrightarrows \mathbf{Y}^\omega$ (parallelization)
- $f \times g : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} \times \mathbf{V}$ (parallel product)
- $f \star g = \max\{f' \circ g' \mid f' \leq_{\mathrm{W}} f \wedge g' \leq_{\mathrm{W}} g\}$ (sequential product)

Operations on Weihrauch degrees

Let $f : \mathbf{X} \rightrightarrows \mathbf{Y}$, $g : \mathbf{U} \rightrightarrows \mathbf{Y}$

- $\widehat{f} : \mathbf{X}^\omega \rightrightarrows \mathbf{Y}^\omega$ (parallelization)
- $f \times g : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} \times \mathbf{V}$ (parallel product)
- $f \star g = \max\{f' \circ g' \mid f' \leq_{\mathrm{W}} f \wedge g' \leq_{\mathrm{W}} g\}$ (sequential product)

Operations on Weihrauch degrees

Let $f : \mathbf{X} \rightrightarrows \mathbf{Y}$, $g : \mathbf{U} \rightrightarrows \mathbf{Y}$

- $\widehat{f} : \mathbf{X}^\omega \rightrightarrows \mathbf{Y}^\omega$ (parallelization)
- $f \times g : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} \times \mathbf{V}$ (parallel product)
- $f \star g = \max\{f' \circ g' \mid f' \leq_{\mathrm{W}} f \wedge g' \leq_{\mathrm{W}} g\}$ (sequential product)

Operations on Weihrauch degrees

Let $f : \mathbf{X} \rightrightarrows \mathbf{Y}$, $g : \mathbf{U} \rightrightarrows \mathbf{Y}$

- $\widehat{f} : \mathbf{X}^\omega \rightrightarrows \mathbf{Y}^\omega$ (parallelization)
- $f \times g : \mathbf{X} \times \mathbf{U} \rightrightarrows \mathbf{Y} \times \mathbf{V}$ (parallel product)
- $f \star g = \max\{f' \circ g' \mid f' \leq_{\mathrm{W}} f \wedge g' \leq_{\mathrm{W}} g\}$ (sequential product)

Some special Weihrauch degrees

Definition
$\mathrm{LPO} : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ defined via $\mathrm{LPO}(0^\omega) = 1$ and $\mathrm{LPO}(p) = 0$ for $p \neq 0^\omega$.

Definition
$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined via $\lim(p)(n) = \lim_{k\to\infty} p(\langle n,k\rangle)$.

Observation
$\lim \equiv_{\mathrm{W}} \widehat{LPO}$

Definition
$\chi_{\Pi^1_1} : \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \to \{0,1\}$ defined via $\chi_{\Pi^1_1}(\emptyset) = 1$ and $\chi_{\Pi^1_1}(A) = 0$ if $A \neq \emptyset$.

Some special Weihrauch degrees

Definition

$\mathrm{LPO} : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ defined via $\mathrm{LPO}(0^\omega) = 1$ and $\mathrm{LPO}(p) = 0$ for $p \neq 0^\omega$.

Definition

$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined via $\lim(p)(n) = \lim_{k\to\infty} p(\langle n,k\rangle)$.

Observation

$\lim \equiv_{\mathrm{W}} \widehat{LPO}$

Definition

$\chi_{\Pi^1_1} : \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \to \{0,1\}$ defined via $\chi_{\Pi^1_1}(\emptyset) = 1$ and $\chi_{\Pi^1_1}(A) = 0$ if $A \neq \emptyset$.

Some special Weihrauch degrees

Definition

$\mathrm{LPO} : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ defined via $\mathrm{LPO}(0^\omega) = 1$ and $\mathrm{LPO}(p) = 0$ for $p \neq 0^\omega$.

Definition

$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined via $\lim(p)(n) = \lim_{k\to\infty} p(\langle n, k\rangle)$.

Observation

$\lim \equiv_{\mathrm{W}} \widehat{LPO}$

Definition

$\chi_{\Pi^1_1} : \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \to \{0,1\}$ defined via $\chi_{\Pi^1_1}(\emptyset) = 1$ and $\chi_{\Pi^1_1}(A) = 0$ if $A \neq \emptyset$.

Some special Weihrauch degrees

Definition

$\mathrm{LPO} : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}$ defined via $\mathrm{LPO}(0^\omega) = 1$ and $\mathrm{LPO}(p) = 0$ for $p \neq 0^\omega$.

Definition

$\lim :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ defined via $\lim(p)(n) = \lim_{k\to\infty} p(\langle n, k\rangle)$.

Observation

$\lim \equiv_{\mathrm{W}} \widehat{LPO}$

Definition

$\chi_{\Pi^1_1} : \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \to \{0, 1\}$ defined via $\chi_{\Pi^1_1}(\emptyset) = 1$ and $\chi_{\Pi^1_1}(A) = 0$ if $A \neq \emptyset$.

Closed Choice

Definition (Closed choice)

$\mathrm{C}_{\mathbf{X}} :\subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}$ defined via $A \in \mathrm{dom}(\mathrm{C}_{\mathbf{X}})$ if $A \neq \emptyset$ and $x \in \mathrm{C}_{\mathbf{X}}(A)$ iff $x \in A$.

Definition (Unique choice)

$\mathrm{UC}_{\mathbf{X}}$ is the restriction of $\mathrm{C}_{\mathbf{X}}$ to singletons.

Closed Choice

Definition (Closed choice)

$\mathrm{C}_{\mathbf{X}} :\subseteq \mathcal{A}(\mathbf{X}) \rightrightarrows \mathbf{X}$ defined via $A \in \mathrm{dom}(\mathrm{C}_{\mathbf{X}})$ if $A \neq \emptyset$ and $x \in \mathrm{C}_{\mathbf{X}}(A)$ iff $x \in A$.

Definition (Unique choice)

$\mathrm{UC}_{\mathbf{X}}$ is the restriction of $\mathrm{C}_{\mathbf{X}}$ to singletons.

The correspondence

Reverse math	Weihrauch degrees
RCA_0	computable
WKL_0	$\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$
ACA_0	$\lim \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \lim \star \ldots \star \lim$
ATR_0	???
$\Pi^1_1\mathrm{-CA}$	$\widehat{\chi_{\Pi^1_1}} \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \widehat{\chi_{\Pi^1_1}} \star \ldots \star \widehat{\chi_{\Pi^1_1}}$

Question (Marcone)

What Weihrauch degree corresponds to ATR_0*?*

The correspondence

Reverse math	Weihrauch degrees
RCA_0	computable
WKL_0	$\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$
ACA_0	$\mathsf{lim} \le_{\mathrm{W}} T \le_{\mathrm{W}} \mathsf{lim} \star \ldots \star \mathsf{lim}$
ATR_0	???
$\Pi^1_1\mathrm{-CA}$	$\widehat{\chi_{\Pi^1_1}} \le_{\mathrm{W}} T \le_{\mathrm{W}} \widehat{\chi_{\Pi^1_1}} \star \ldots \star \widehat{\chi_{\Pi^1_1}}$

Question (Marcone)

What Weihrauch degree corresponds to ATR_0*?*

The correspondence

Reverse math	Weihrauch degrees
RCA_0	computable
WKL_0	$\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$
ACA_0	$\lim \leq_W T \leq_W \lim \star \ldots \star \lim$
ATR_0	???
$\Pi^1_1\mathrm{-CA}$	$\widehat{\chi_{\Pi^1_1}} \leq_W T \leq_W \widehat{\chi_{\Pi^1_1}} \star \ldots \star \widehat{\chi_{\Pi^1_1}}$

Question (Marcone)

What Weihrauch degree corresponds to ATR_0*?*

The correspondence

Reverse math	Weihrauch degrees
RCA_0	computable
WKL_0	$\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$
ACA_0	$\lim \leq_W T \leq_W \lim \star \ldots \star \lim$
ATR_0	???
$\Pi^1_1{-}\mathrm{CA}$	$\widehat{\chi_{\Pi^1_1}} \leq_W T \leq_W \widehat{\chi_{\Pi^1_1}} \star \ldots \star \widehat{\chi_{\Pi^1_1}}$

Question (Marcone)

What Weihrauch degree corresponds to ATR_0*?*

The correspondence

Reverse math	Weihrauch degrees
RCA_0	computable
WKL_0	$\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$
ACA_0	$\lim \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \lim \star \ldots \star \lim$
ATR_0	???
$\Pi^1_1\text{–}\mathrm{CA}$	$\widehat{\chi_{\Pi^1_1}} \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \widehat{\chi_{\Pi^1_1}} \star \ldots \star \widehat{\chi_{\Pi^1_1}}$

Question (Marcone)

What Weihrauch degree corresponds to ATR_0*?*

The correspondence

Reverse math	Weihrauch degrees
RCA_0	computable
WKL_0	$\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$
ACA_0	$\lim \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \lim \star \ldots \star \lim$
ATR_0	???
$\Pi^1_1\mathrm{-CA}$	$\widehat{\chi_{\Pi^1_1}} \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \widehat{\chi_{\Pi^1_1}} \star \ldots \star \widehat{\chi_{\Pi^1_1}}$

Question (Marcone)

What Weihrauch degree corresponds to ATR_0?

The correspondence

Reverse math	Weihrauch degrees
RCA_0	computable
WKL_0	$\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$
ACA_0	$\lim \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \lim \star \ldots \star \lim$
ATR_0	???
$\Pi^1_1\mathrm{-CA}$	$\widehat{\chi_{\Pi^1_1}} \leq_{\mathrm{W}} T \leq_{\mathrm{W}} \widehat{\chi_{\Pi^1_1}} \star \ldots \star \widehat{\chi_{\Pi^1_1}}$

Question (Marcone)

What Weihrauch degree corresponds to ATR_0*?*

Outline

The theorems

The context

The results

The problems I

Definition

Let $\mathrm{FindWS}_\Sigma : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game where Player 1 has a winning strategy to a winning strategy (for Player 1).

Definition

Let $\mathrm{FindWS}_\Pi : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game where Player 2 has a winning strategy to a winning strategy (for Player 2).

Definition

Let $\mathrm{Det}_\Sigma : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game to a pair of strategies (σ, τ) such that either σ is winning for Player 1 or τ is winning for Player 2.

The problems I

Definition

Let $\mathrm{FindWS}_\Sigma : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game where Player 1 has a winning strategy to a winning strategy (for Player 1).

Definition

Let $\mathrm{FindWS}_\Pi : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game where Player 2 has a winning strategy to a winning strategy (for Player 2).

Definition

Let $\mathrm{Det}_\Sigma : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game to a pair of strategies (σ, τ) such that either σ is winning for Player 1 or τ is winning for Player 2.

The problems I

Definition

Let $\mathrm{FindWS}_\Sigma : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game where Player 1 has a winning strategy to a winning strategy (for Player 1).

Definition

Let $\mathrm{FindWS}_\Pi : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game where Player 2 has a winning strategy to a winning strategy (for Player 2).

Definition

Let $\mathrm{Det}_\Sigma : \mathcal{O}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ map a Σ^0_1-game to a pair of strategies (σ, τ) such that either σ is winning for Player 1 or τ is winning for Player 2.

The problems II

Definition

Let $\mathrm{List} :\subseteq \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \rightrightarrows \mathbb{N}^{\mathbb{N}}$ map $\emptyset$ to 0^ω and countable non-empty A to some $\langle q_0, q_1, \ldots \rangle$ such that $A = \{q_i \mid i \in \mathbb{N}\}$.

Definition

Let $\mathrm{PTT}_1 :\subseteq \mathrm{Trees} \rightrightarrows \mathrm{Trees}$ map a tree T such that $[T]$ is uncountable to some perfect subtree.

Definition

Let $\mathrm{PTT}_2 : \mathrm{Trees} \rightrightarrows \mathbb{N}^{\mathbb{N}} \times \mathrm{Trees}$ map a tree T to a pair (p, S) such that either $p \in \mathrm{List}([T])$ or $S \in \mathrm{PTT}_1(S)$.

The problems II

Definition

Let $\mathrm{List} :\subseteq \mathcal{A}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map $\emptyset$ to 0^ω and countable non-empty A to some $\langle q_0, q_1, \ldots \rangle$ such that $A = \{q_i \mid i \in \mathbb{N}\}$.

Definition

Let $\mathrm{PTT}_1 :\subseteq \mathrm{Trees} \rightrightarrows \mathrm{Trees}$ map a tree T such that $[T]$ is uncountable to some perfect subtree.

Definition

Let $\mathrm{PTT}_2 : \mathrm{Trees} \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathrm{Trees}$ map a tree T to a pair (p, S) such that either $p \in \mathrm{List}([T])$ or $S \in \mathrm{PTT}_1(S)$.

The problems II

Definition

Let $\mathrm{List} :\subseteq \mathcal{A}(\mathbb{N}^\mathbb{N}) \rightrightarrows \mathbb{N}^\mathbb{N}$ map $\emptyset$ to 0^ω and countable non-empty A to some $\langle q_0, q_1, \ldots \rangle$ such that $A = \{q_i \mid i \in \mathbb{N}\}$.

Definition

Let $\mathrm{PTT}_1 :\subseteq \mathrm{Trees} \rightrightarrows \mathrm{Trees}$ map a tree T such that $[T]$ is uncountable to some perfect subtree.

Definition

Let $\mathrm{PTT}_2 : \mathrm{Trees} \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathrm{Trees}$ map a tree T to a pair (p, S) such that either $p \in \mathrm{List}([T])$ or $S \in \mathrm{PTT}_1(S)$.

Classifications

Theorem

$UC_{\mathbb{N}^\mathbb{N}} \equiv_W \mathrm{FindWS}_\Sigma \equiv_W \mathrm{List}$.

Theorem

$C_{\mathbb{N}^\mathbb{N}} \equiv_W \mathrm{FindWS}_\Pi \equiv_W \mathrm{PTT}_1$.

($C_{\mathbb{N}^\mathbb{N}} \equiv_W \mathrm{PTT}_1$ due to Brattka and Marcone).

Classifications

Theorem

$UC_{\mathbb{N}^\mathbb{N}} \equiv_W \mathrm{FindWS}_\Sigma \equiv_W \mathrm{List}$.

Theorem

$C_{\mathbb{N}^\mathbb{N}} \equiv_W \mathrm{FindWS}_\Pi \equiv_W \mathrm{PTT}_1$.

($\mathrm{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathrm{PTT}_1$ due to Brattka and Marcone).

Classifications II

Observation

$\chi_{\Pi^1_1} \leq_W UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{Det}_\Sigma$ *and* $\chi_{\Pi^1_1} \leq_W UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{PTT}_2$

Corollary

$\mathrm{Det}_\Sigma \not\leq_W C_{\mathbb{N}^\mathbb{N}}$ *and* $\mathrm{PTT}_2 \not\leq_W C_{\mathbb{N}^\mathbb{N}}$.

Corollary

$\mathrm{Det}_\Sigma \leq_W C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi^1_1}$ *and* $\mathrm{PTT}_2 \leq_W C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi^1_1}$.

Proposition

$\mathrm{Det}_\Sigma \times \mathrm{Det}_\Sigma \not\leq_W \mathrm{Det}_\Sigma$ *and* $\mathrm{PTT}_2 \times \mathrm{PTT}_2 \not\leq_W \mathrm{PTT}_2$.

Classifications II

Observation

$\chi_{\Pi^1_1} \leq_W UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{Det}_\Sigma$ *and* $\chi_{\Pi^1_1} \leq_W UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{PTT}_2$

Corollary

$\mathrm{Det}_\Sigma \nleq_W C_{\mathbb{N}^\mathbb{N}}$ *and* $\mathrm{PTT}_2 \nleq_W C_{\mathbb{N}^\mathbb{N}}$.

Corollary

$\mathrm{Det}_\Sigma \leq_W C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi^1_1}$ *and* $\mathrm{PTT}_2 \leq_W C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi^1_1}$.

Proposition

$\mathrm{Det}_\Sigma \times \mathrm{Det}_\Sigma \nleq_W \mathrm{Det}_\Sigma$ *and* $\mathrm{PTT}_2 \times \mathrm{PTT}_2 \nleq_W \mathrm{PTT}_2$.

Classifications II

Observation

$\chi_{\Pi_1^1} \leq_{\mathrm{W}} UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{Det}_\Sigma$ *and* $\chi_{\Pi_1^1} \leq_{\mathrm{W}} UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{PTT}_2$

Corollary

$\mathrm{Det}_\Sigma \not\leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}}$ *and* $\mathrm{PTT}_2 \not\leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}}$.

Corollary

$\mathrm{Det}_\Sigma \leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi_1^1}$ *and* $\mathrm{PTT}_2 \leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi_1^1}$.

Proposition

$\mathrm{Det}_\Sigma \times \mathrm{Det}_\Sigma \not\leq_{\mathrm{W}} \mathrm{Det}_\Sigma$ *and* $\mathrm{PTT}_2 \times \mathrm{PTT}_2 \not\leq_{\mathrm{W}} \mathrm{PTT}_2$.

Classifications II

Observation

$\chi_{\Pi^1_1} \leq_{\mathrm{W}} UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{Det}_\Sigma$ *and* $\chi_{\Pi^1_1} \leq_{\mathrm{W}} UC_{\mathbb{N}^\mathbb{N}} \star \mathrm{PTT}_2$

Corollary

$\mathrm{Det}_\Sigma \not\leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}}$ *and* $\mathrm{PTT}_2 \not\leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}}$.

Corollary

$\mathrm{Det}_\Sigma \leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi^1_1}$ *and* $\mathrm{PTT}_2 \leq_{\mathrm{W}} C_{\mathbb{N}^\mathbb{N}} \star \chi_{\Pi^1_1}$.

Proposition

$\mathrm{Det}_\Sigma \times \mathrm{Det}_\Sigma \not\leq_{\mathrm{W}} \mathrm{Det}_\Sigma$ *and* $\mathrm{PTT}_2 \times \mathrm{PTT}_2 \not\leq_{\mathrm{W}} \mathrm{PTT}_2$.

The big question I

Question

Does $\mathrm{Det}_\Sigma \equiv_{\mathrm{W}} \mathrm{PTT}_2$ *hold?*

An attempt

Definition

Let $t : \mathbf{Z} \to \{0, 1\}$, $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $g :\subseteq \mathbf{A} \rightrightarrows \mathbf{B}$ with $\mathbf{X}, \mathbf{Y}, \mathbf{A}, \mathbf{B}$ being precomplete. Define

$$h := [\text{if } t \text{ then } f \text{ else } g] :\subseteq \mathbf{Z} \times \mathbf{X} \times \mathbf{A} \rightrightarrows \mathbf{Y} \rightrightarrows \mathbf{B}$$

via $(z, x, a) \in \mathrm{dom}(h)$ if $t(z) = 1$ and $x \in \mathrm{dom}(f)$ or $t(z) = 0$ and $a \in \mathrm{dom}(g)$, and $(y, b) \in h(z, x, a)$ if $t(z) = 1$ and $y \in f(x)$ or $t(z) = 0$ and $b \in g(a)$.

The big question II

Observation

$\mathrm{Det}_\Sigma \leq_\mathrm{W} [\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}]$ *and*
$\mathrm{PTT}_2 \leq_\mathrm{W} [\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}]$

Question

Does $[\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}] \leq_\mathrm{W} \mathrm{Det}_\Sigma$ *and/or*
$[\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}] \leq_\mathrm{W} \mathrm{PTT}_2$ *hold?*

The big question II

Observation

$\mathrm{Det}_\Sigma \leq_\mathrm{W} [\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}]$ *and*
$\mathrm{PTT}_2 \leq_\mathrm{W} [\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}]$

Question

Does $[\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}] \leq_\mathrm{W} \mathrm{Det}_\Sigma$ *and/or*
$[\text{if } \chi_{\Pi^1_1} \text{ then } UC_{\mathbb{N}^\mathbb{N}} \text{ else } C_{\mathbb{N}^\mathbb{N}}] \leq_\mathrm{W} \mathrm{PTT}_2$ *hold?*