Fractal Intersections and Products via Algorithmic Dimension

Neil Lutz
Rutgers University

June 26, 2017

Goal:

Use algorithmic information theory to answer fundamental questions in fractal geometry.

Agenda:

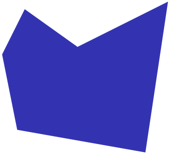
- Discuss classical and algorithmic notions of dimension.
- Describe a recent point-to-set principle that relates them.
- Describe a notion of conditional dimension.
- Apply these new tools bound the classical dimension of products and slices of fractals.
  - Special case of intersections — one of the sets is a vertical line.

What is dimension?

Informally, it's the number of free parameters: The number of parameters needed to specify an arbitrary element inside a set given a description for the set.

What is dimension?

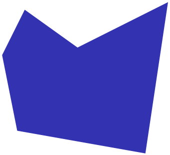
Informally, it's the number of free parameters: The number of parameters needed to specify an arbitrary element inside a set given a description for the set.



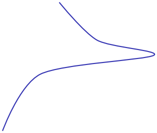
2

What is dimension?

Informally, it's the number of free parameters: The number of parameters needed to specify an arbitrary element inside a set given a description for the set.



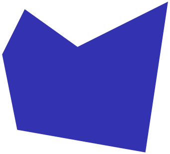
2



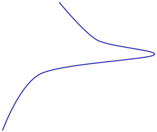
1

What is dimension?

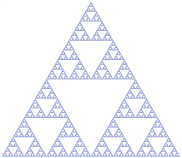
Informally, it's the number of free parameters: The number of parameters needed to specify an arbitrary element inside a set given a description for the set.



2



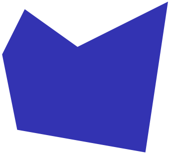
1



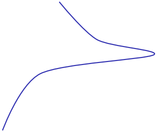
???

What is dimension?

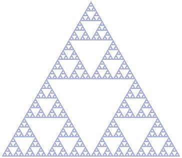
Informally, it's the number of free parameters: The number of parameters needed to specify an arbitrary element inside a set given a description for the set.



2



1

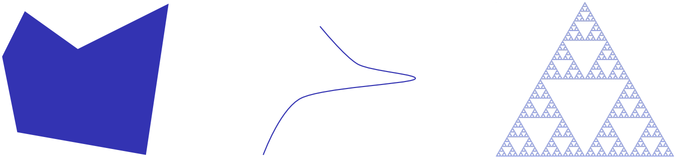


???

We want a way to quantitatively classify sets of measure zero.

What is dimension?

Informally, it's the number of free parameters: The number of parameters needed to specify an arbitrary element inside a set given a description for the set.



2 1 ???

We want a way to quantitatively classify sets of measure zero.

Example: Suppose an algorithm succeeds with probability 1 but fails in the worst case. How much control does an adversary need to have over the environment to ensure failure?

Fractal Dimension: Measure Theoretic Approach

How strongly does granularity affect measurement of the set?

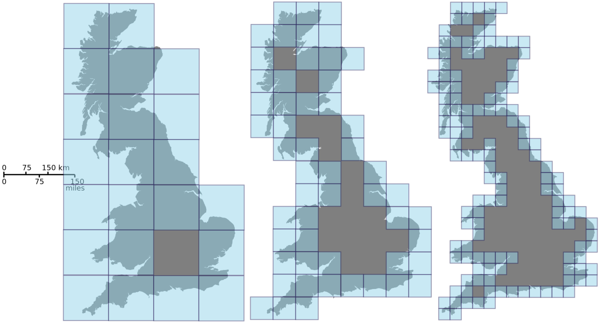


Image credit: Alexis Monnerot-Dumaine

Let N_ε = number of boxes with side ε needed to cover the set.

Fractal Dimension: Measure Theoretic Approach

How strongly does granularity affect measurement of the set?

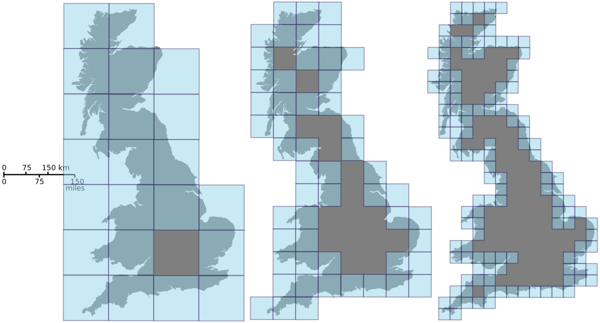


Image credit: Alexis Monnerot-Dumaine

Let N_ε = number of boxes with side ε needed to cover the set.

Consider $\lim_{\varepsilon \to 0} N_\varepsilon \cdot \varepsilon^s$.

Fractal Dimension: Measure Theoretic Approach

How strongly does granularity affect measurement of the set?

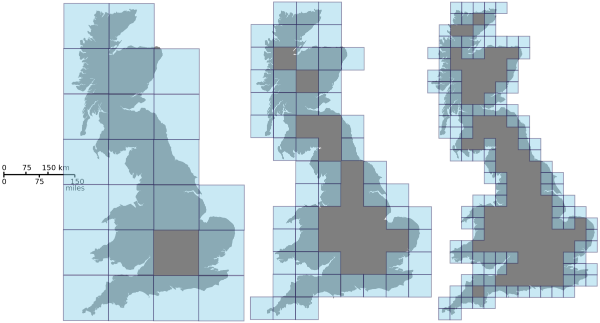


Image credit: Alexis Monnerot-Dumaine

Let N_ε = number of boxes with side ε needed to cover the set.

Consider $\lim_{\varepsilon \to 0} N_\varepsilon \cdot \varepsilon^s$.
Infinite for $s = 1$ (infinite length) and 0 for $s = 2$ (0 area).

Fractal Dimension: Measure Theoretic Approach

How strongly does granularity affect measurement of the set?

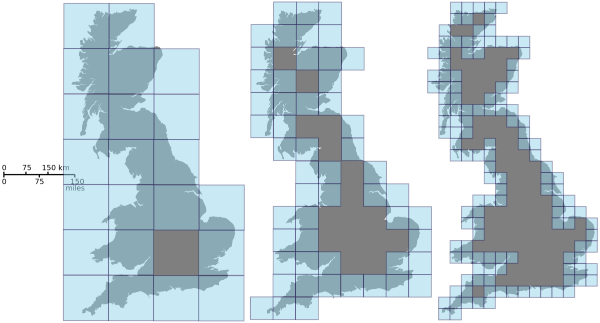


Image credit: Alexis Monnerot-Dumaine

Let N_ε = number of boxes with side ε needed to cover the set.

Consider $\lim_{\varepsilon \to 0} N_\varepsilon \cdot \varepsilon^s$.
Infinite for $s = 1$ (infinite length) and 0 for $s = 2$ (0 area).
In fact, the limit is positive and finite for <u>at most</u> one value of s.

Hausdorff Dimension

The most standard, robust notion of fractal dimension.

Hausdorff Dimension

The most standard, robust notion of fractal dimension.

$H^s(E) = s$-dimensional Hausdorff measure of a set $E \subseteq \mathbb{R}^n$.
(Generalizes integer-dimensional Lebesgue outer measure)

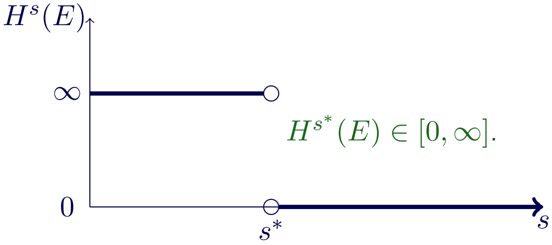
Hausdorff Dimension

The most standard, robust notion of fractal dimension.

$H^s(E) = s$-dimensional Hausdorff measure of a set $E \subseteq \mathbb{R}^n$.
(Generalizes integer-dimensional Lebesgue outer measure)

Hausdorff 1919: The Hausdorff dimension of E is

$$\dim_H(E) = \inf\{s : H^s(E) = 0\} .$$



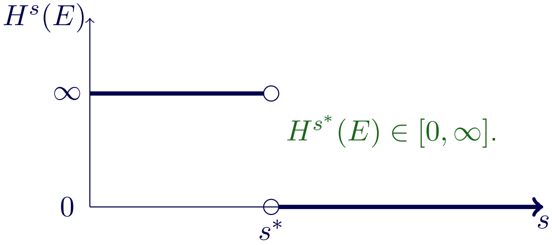
Hausdorff Dimension

The most standard, robust notion of fractal dimension.

$H^s(E)$ = s-dimensional Hausdorff measure of a set $E \subseteq \mathbb{R}^n$.
(Generalizes integer-dimensional Lebesgue outer measure)

Hausdorff 1919: The Hausdorff dimension of E is

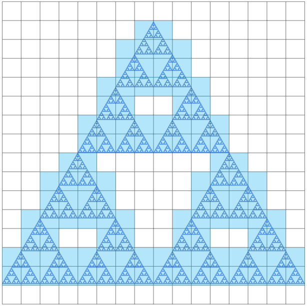
$$\dim_H(E) = \inf\{s : H^s(E) = 0\}\,.$$



It is often difficult to prove lower bounds on $\dim_H(E)$.

Example: Dimension of the Sierpinski triangle

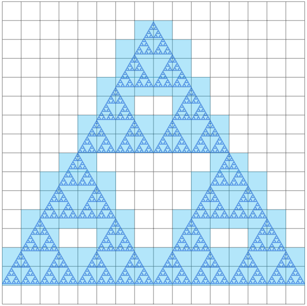
Convenient fact: This set has Hausdorff dimension equal to its box-counting dimension.



$$N_\varepsilon = \theta(\varepsilon^{-\log 3})$$

Example: Dimension of the Sierpinski triangle

Convenient fact: This set has Hausdorff dimension equal to its box-counting dimension.

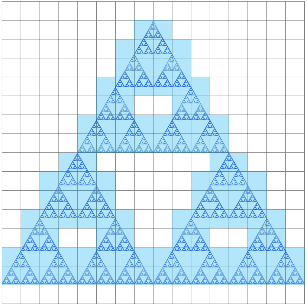


$$N_\varepsilon = \theta(\varepsilon^{-\log 3})$$

$\lim_{\varepsilon \to 0} N_\varepsilon \cdot \varepsilon^s$ can only be positive and finite for $s = \log 3$,
so the Sierpinski triangle has Hausdorff dimension $\log 3 \approx 1.585$.

Example: Dimension of the Sierpinski triangle

Convenient fact: This set has Hausdorff dimension equal to its box-counting dimension.

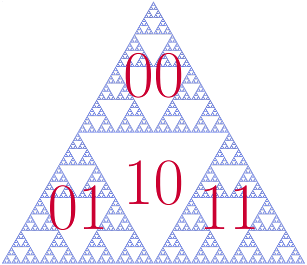


$$N_\varepsilon = \theta(\varepsilon^{-\log 3})$$

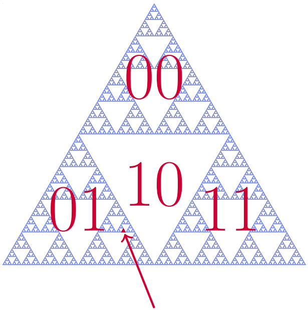
$\lim_{\varepsilon \to 0} N_\varepsilon \cdot \varepsilon^s$ can only be positive and finite for $s = \log 3$,
so the Sierpinski triangle has Hausdorff dimension $\log 3 \approx 1.585$.

In what sense is this the number of free parameters?

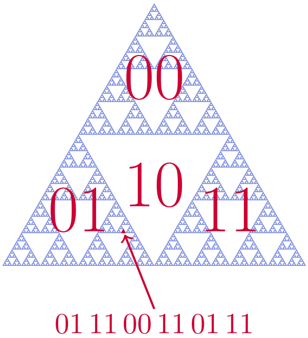
Example: Dimension of the Sierpinski triangle



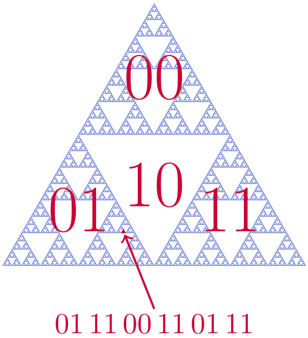
Example: Dimension of the Sierpinski triangle



Example: Dimension of the Sierpinski triangle

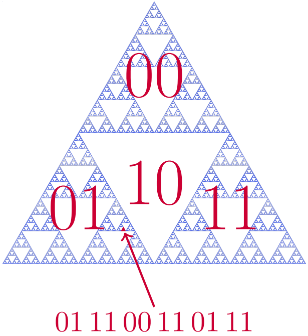


Example: Dimension of the Sierpinski triangle



We can think of the first bit and second bit at each recursion level as two parameters. $2r$ bits approximate a point within $\approx 2^{-r}$ error.

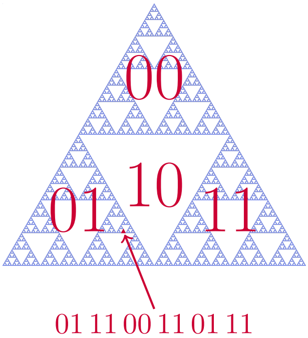
Example: Dimension of the Sierpinski triangle



We can think of the first bit and second bit at each recursion level as two parameters. $2r$ bits approximate a point within $\approx 2^{-r}$ error.

But for points within the fractal set, these parameters are not independent of each other. The $2r$ bits are compressible as data to length $\approx r \log 3$.

Example: Dimension of the Sierpinski triangle



We can think of the first bit and second bit at each recursion level as two parameters. $2r$ bits approximate a point within $\approx 2^{-r}$ error.

But for points within the fractal set, these parameters are not independent of each other. The $2r$ bits are compressible as data to length $\approx r \log 3$.

In this sense, we only need $\log 3 \approx 1.585$ parameters to specify a point within the set.

Algorithmic Information in Bit Strings

We need a formal notion of compressibility:

The Kolmogorov complexity of a bit string $\sigma \in \{0,1\}^*$ is the length of the shortest binary program that outputs σ:

$$K(\sigma) = \min \{ |\pi| : U(\pi) = \sigma \},$$

where U is a universal Turing machine.

Algorithmic Information in Bit Strings

We need a formal notion of compressibility:

The Kolmogorov complexity of a bit string $\sigma \in \{0,1\}^*$ is the length of the shortest binary program that outputs σ:

$$K(\sigma) = \min\{|\pi| : U(\pi) = \sigma\},$$

where U is a universal Turing machine.

- It matters little which U is chosen for this.

Algorithmic Information in Bit Strings

We need a formal notion of compressibility:

The Kolmogorov complexity of a bit string $\sigma \in \{0,1\}^*$ is the length of the shortest binary program that outputs σ:

$$K(\sigma) = \min\{|\pi| : U(\pi) = \sigma\},$$

where U is a universal Turing machine.

- It matters little which U is chosen for this.
- $K(\sigma) =$ amount of algorithmic information in σ.

Algorithmic Information in Bit Strings

We need a formal notion of compressibility:

The Kolmogorov complexity of a bit string $\sigma \in \{0,1\}^*$ is the length of the shortest binary program that outputs σ:

$$K(\sigma) = \min \{ |\pi| : U(\pi) = \sigma \} ,$$

where U is a universal Turing machine.

- It matters little which U is chosen for this.
- $K(\sigma) =$ amount of algorithmic information in σ.
- $K(\sigma) \le |\sigma| + o(|\sigma|)$.

Algorithmic Information in Bit Strings

We need a formal notion of compressibility:

The Kolmogorov complexity of a bit string $\sigma \in \{0,1\}^*$ is the length of the shortest binary program that outputs σ:

$$K(\sigma) = \min\{|\pi| : U(\pi) = \sigma\},$$

where U is a universal Turing machine.

- It matters little which U is chosen for this.
- $K(\sigma)$ = amount of algorithmic information in σ.
- $K(\sigma) \leq |\sigma| + o(|\sigma|)$.
- Extends naturally to other finite data objects
  - e.g., points in $\mathbb{Q}^n$

Algorithmic Information in Euclidean Spaces

Points in $\mathbb{R}^n$ are infinite data objects.

Algorithmic Information in Euclidean Spaces

Points in $\mathbb{R}^n$ are infinite data objects.

The Kolmogorov complexity of a set $E \subseteq \mathbb{Q}^n$ is

$$K(E) = \min\{K(q) : q \in E\}.$$

(Shen and Vereschagin 2002)

Algorithmic Information in Euclidean Spaces

Points in $\mathbb{R}^n$ are infinite data objects.

The Kolmogorov complexity of a set $E \subseteq \mathbb{Q}^n$ is

$$K(E) = \min\{K(q) : q \in E\}\,.$$

(Shen and Vereschagin 2002)

The Kolmogorov complexity of a set $E \subseteq \mathbb{R}^n$ is

$$K(E) = K(E \cap \mathbb{Q}^n)\,.$$

Algorithmic Information in Euclidean Spaces

Points in $\mathbb{R}^n$ are infinite data objects.

The Kolmogorov complexity of a set $E \subseteq \mathbb{Q}^n$ is

$$K(E) = \min\{K(q) : q \in E\}\,.$$

(Shen and Vereschagin 2002)

The Kolmogorov complexity of a set $E \subseteq \mathbb{R}^n$ is

$$K(E) = K(E \cap \mathbb{Q}^n)\,.$$

Note that

$$E \subseteq F \Rightarrow K(E) \geq K(F)\,.$$

Algorithmic Information in Euclidean Spaces

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$. The Kolmogorov complexity of x at precision r is

$$K_r(x) = K\big(B_{2^{-r}}(x)\big)\ ,$$

i.e., the number of bits required to specify some rational point $q \in \mathbb{Q}^n$ such that $|q - x| \leq 2^{-r}$.

Algorithmic Information in Euclidean Spaces

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$. The Kolmogorov complexity of x at precision r is

$$K_r(x) = K\left(B_{2^{-r}}(x)\right),$$

i.e., the number of bits required to specify some rational point $q \in \mathbb{Q}^n$ such that $|q - x| \leq 2^{-r}$.

We say x is (algorithmically) random if $K_r(x) \geq nr - O(1)$.

Fact: Almost all points are random.

Algorithmic Dimension

At precision r, $x \in \mathbb{R}^n$ has information density

$$0 \leq \frac{K_r(x)}{r} \leq n + o(1)\,.$$

Algorithmic Dimension

At precision r, $x \in \mathbb{R}^n$ has information density

$$0 \leq \frac{K_r(x)}{r} \leq n + o(1)\,.$$

J. Lutz and Mayordomo: The algorithmic dimension of $x \in \mathbb{R}^n$ is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}\,.$$

Algorithmic Dimension

At precision r, $x \in \mathbb{R}^n$ has information density

$$0 \leq \frac{K_r(x)}{r} \leq n + o(1)\,.$$

J. Lutz and Mayordomo: The algorithmic dimension of $x \in \mathbb{R}^n$ is

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}\,.$$

Examples:

- If x is computable, then there is a finite program that outputs x precisely, so $K_r(x) = O(1)$ and $\dim(x) = 0$.

Algorithmic Dimension

At precision r, $x \in \mathbb{R}^n$ has information density

$$0 \leq \frac{K_r(x)}{r} \leq n + o(1)\,.$$

J. Lutz and Mayordomo: The algorithmic dimension of $x \in \mathbb{R}^n$ is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}\,.$$

Examples:

- If x is computable, then there is a finite program that outputs x precisely, so $K_r(x) = O(1)$ and $\dim(x) = 0$.
- If $x \in \mathbb{R}^n$ is random, then

$$nr - O(1) \leq K_r(x) \leq nr + o(r)\,,$$

  so $\dim(x) = n$.

Algorithmic Dimension

At precision r, $x \in \mathbb{R}^n$ has information density

$$0 \leq \frac{K_r(x)}{r} \leq n + o(1)\,.$$

J. Lutz and Mayordomo: The algorithmic dimension of $x \in \mathbb{R}^n$ is

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}\,.$$

Examples:

- If x is computable, then there is a finite program that outputs x precisely, so $K_r(x) = O(1)$ and $\dim(x) = 0$.
- If $x \in \mathbb{R}^n$ is random, then

  $$nr - O(1) \leq K_r(x) \leq nr + o(r)\,,$$

  so $\dim(x) = n$.
- The converse does not hold in either case.

Aren't points supposed to have dimension 0?

For the Sierpinski triangle T, we have

$$\dim_H(T) = \sup_{x \in T} \dim(x) \, .$$

Aren't points supposed to have dimension 0?

For the Sierpinski triangle T, we have

$$\dim_H(T) = \sup_{x \in T} \dim(x)\,.$$

This relationship does not hold in general: Consider the singleton $\{y\}$, where $y \in \mathbb{R}^n$ is random. Then $\dim_H(\{y\}) = 0$, but

$$\sup_{x \in \{y\}} \dim(x) = \dim(y) = n\,.$$

Aren't points supposed to have dimension 0?

For the Sierpinski triangle T, we have

$$\dim_H(T) = \sup_{x \in T} \dim(x)\,.$$

This relationship does not hold in general: Consider the singleton $\{y\}$, where $y \in \mathbb{R}^n$ is random. Then $\dim_H(\{y\}) = 0$, but

$$\sup_{x \in \{y\}} \dim(x) = \dim(y) = n\,.$$

But we said dimension is the number of free parameters needed to specify a point given a description of the set.

The universal machine reading our program to estimate $x \in E$ ought to have access to a description of E.

Relative Dimension

The Kolmogorov complexity of a bitstring $\sigma \in \{0,1\}^*$ relative to an oracle $w \in \{0,1\}^\infty$ is

$$K^w(\sigma) = \min\left\{|\pi| : U^w(\pi) = \sigma\right\},$$

where U is a universal oracle machine: It can query any bit of w as a computational step.

Relative Dimension

The Kolmogorov complexity of a bitstring $\sigma \in \{0,1\}^*$ relative to an oracle $w \in \{0,1\}^\infty$ is

$$K^w(\sigma) = \min\left\{|\pi| : U^w(\pi) = \sigma\right\},$$

where U is a universal oracle machine: It can query any bit of w as a computational step.

Relative Dimension

The Kolmogorov complexity of a bitstring $\sigma \in \{0,1\}^*$ relative to an oracle $w \in \{0,1\}^\infty$ is

$$K^w(\sigma) = \min\left\{|\pi| : U^w(\pi) = \sigma\right\},$$

where U is a universal oracle machine: It can query any bit of w as a computational step.

The dimension of a point $x \in \mathbb{R}^n$ relative to oracle w is

$$\dim^w(x) = \liminf_{r\to\infty} \frac{K^w_r(x)}{r}.$$

Relative Dimension

The Kolmogorov complexity of a bitstring $\sigma \in \{0,1\}^*$ relative to an oracle $w \in \{0,1\}^\infty$ is

$$K^w(\sigma) = \min\left\{|\pi| : U^w(\pi) = \sigma\right\},$$

where U is a universal oracle machine: It can query any bit of w as a computational step.

The dimension of a point $x \in \mathbb{R}^n$ relative to oracle w is

$$\dim^w(x) = \liminf_{r \to \infty} \frac{K_r^w(x)}{r}.$$

- Note that the oracle can encode a point in $\mathbb{R}^n$.

Relative Dimension

The Kolmogorov complexity of a bitstring $\sigma \in \{0,1\}^*$ relative to an oracle $w \in \{0,1\}^\infty$ is

$$K^w(\sigma) = \min\left\{|\pi| : U^w(\pi) = \sigma\right\},$$

where U is a universal oracle machine: It can query any bit of w as a computational step.

The dimension of a point $x \in \mathbb{R}^n$ relative to oracle w is

$$\dim^w(x) = \liminf_{r \to \infty} \frac{K_r^w(x)}{r}.$$

- Note that the oracle can encode a point in $\mathbb{R}^n$.
- For all $x \in \mathbb{R}^n$, $\dim^x(x) = 0$.

Point-to-Set Principle (Lutz & Lutz '17)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_w \sup_{x \in E} \dim^w(x)\,.$$

Point-to-Set Principle (Lutz & Lutz '17)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_w \sup_{x \in E} \dim^w(x) \,.$$

classical Hausdorff dimension

dimensions of individual points

Point-to-Set Principle (Lutz & Lutz '17)

For every set $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_w \sup_{x \in E} \dim^w(x)\,.$$

classical Hausdorff dimension

dimensions of individual points

$\therefore$ In order to prove a lower bound

$$\dim_H(E) \geq \alpha\,,$$

it is enough to show that for every oracle w and $\varepsilon > 0$, there is some point $x \in E$ with

$$\dim^w(x) \geq \alpha - \varepsilon\,.$$

Conditional Dimension

The conditional Kolomogorov complexity of $p \in \mathbb{Q}^m$ given $q \in \mathbb{Q}^n$:

$$K(p|q) = \min\left\{|\pi| : \pi \in \{0,1\}^* \text{ and } U(\pi, q) = p\right\}.$$

Conditional Dimension

The conditional Kolomogorov complexity of $p \in \mathbb{Q}^m$ given $q \in \mathbb{Q}^n$:

$$K(p|q) = \min\left\{|\pi| : \pi \in \{0,1\}^* \text{ and } U(\pi, q) = p\right\}.$$

The conditional Kolmogorov complexity of $E \subseteq \mathbb{Q}^m$ given $F \subseteq \mathbb{Q}^n$:

$$K(E|F) = \max_{q \in F} \min_{p \in E} K(p|q).$$

Conditional Dimension

The conditional Kolomogorov complexity of $p \in \mathbb{Q}^m$ given $q \in \mathbb{Q}^n$:

$$K(p|q) = \min \{|\pi| : \pi \in \{0,1\}^* \text{ and } U(\pi, q) = p\}.$$

The conditional Kolmogorov complexity of $E \subseteq \mathbb{Q}^m$ given $F \subseteq \mathbb{Q}^n$:

$$K(E|F) = \max_{q \in F} \min_{p \in E} K(p|q).$$

The conditional Kolmogorov complexity of $x \in \mathbb{R}^m$ at precision y given $y \in \mathbb{R}^n$ at precision s:

$$K_{r,s}(x|y) = K(B_{2^{-r}}(x)|B_{2^{-s}}(y)).$$

Conditional Dimension

Definition (Lutz & Lutz '17)

The conditional dimension of $x \in \mathbb{R}^m$ given $y \in \mathbb{R}^n$ is

$$\dim(x|y) = \liminf_{r \to \infty} \frac{K_{r,r}(x|y)}{r} \, .$$

Conditional Dimension

Definition (Lutz & Lutz '17)

The conditional dimension of $x \in \mathbb{R}^m$ given $y \in \mathbb{R}^n$ is

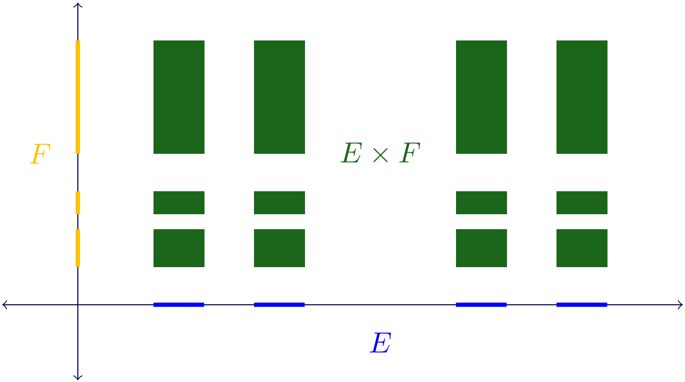
$$\dim(x|y) = \liminf_{r \to \infty} \frac{K_{r,r}(x|y)}{r} .$$

- Obeys a chain rule: $\dim(x, y) \geq \dim(x|y) + \dim(y)$.
- Bounded below by relative dimension: $\dim(x|y) \geq \dim^y(x)$.

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F) \,.$$



Easy for Borel sets. Was significantly more difficult for general sets.

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E \times F) = \sup_{(x,y)\in E\times F} \dim^w(x,y)\,,$$

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E \times F) = \sup_{(x,y) \in E \times F} \dim^w(x,y)\,,$$

and for every $\varepsilon > 0$ there exist $x \in E$ and $y \in F$ such that

$$\dim^w(x) \geq \dim_H(E) - \varepsilon \quad \text{and} \quad \dim^{w,x}(y) \geq \dim_H(F) - \varepsilon\,.$$

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E \times F) = \sup_{(x,y)\in E\times F} \dim^w(x,y)\,,$$

and for every $\varepsilon > 0$ there exist $x \in E$ and $y \in F$ such that

$$\dim^w(x) \geq \dim_H(E) - \varepsilon \quad \text{and} \quad \dim^{w,x}(y) \geq \dim_H(F) - \varepsilon\,.$$

For this x and y,

$$\dim_H(E \times F) \geq \dim^w(x,y)$$

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E \times F) = \sup_{(x,y) \in E \times F} \dim^w(x, y)\,,$$

and for every $\varepsilon > 0$ there exist $x \in E$ and $y \in F$ such that

$$\dim^w(x) \geq \dim_H(E) - \varepsilon \quad \text{and} \quad \dim^{w,x}(y) \geq \dim_H(F) - \varepsilon\,.$$

For this x and y,

$$\begin{aligned} \dim_H(E \times F) &\geq \dim^w(x, y) \\ &\geq \dim^w(x) + \dim^w(y|x) \end{aligned}$$

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E \times F) = \sup_{(x,y)\in E\times F} \dim^w(x,y)\,,$$

and for every $\varepsilon > 0$ there exist $x \in E$ and $y \in F$ such that

$$\dim^w(x) \geq \dim_H(E) - \varepsilon \quad \text{and} \quad \dim^{w,x}(y) \geq \dim_H(F) - \varepsilon\,.$$

For this x and y,

$$\begin{aligned} \dim_H(E \times F) &\geq \dim^w(x,y) \\ &\geq \dim^w(x) + \dim^w(y|x) \\ &\geq \dim^w(x) + \dim^{w,x}(y) \end{aligned}$$

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E \times F) = \sup_{(x,y)\in E\times F} \dim^w(x,y)\,,$$

and for every $\varepsilon > 0$ there exist $x \in E$ and $y \in F$ such that

$$\dim^w(x) \geq \dim_H(E) - \varepsilon \quad \text{and} \quad \dim^{w,x}(y) \geq \dim_H(F) - \varepsilon\,.$$

For this x and y,

$$\begin{aligned} \dim_H(E \times F) &\geq \dim^w(x,y) \\ &\geq \dim^w(x) + \dim^w(y|x) \\ &\geq \dim^w(x) + \dim^{w,x}(y) \\ &\geq \dim_H(E) + \dim_H(F) - 2\varepsilon\,. \end{aligned}$$

Product Theorem (Marstrand 1954)

For all $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F)\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E \times F) = \sup_{(x,y)\in E\times F} \dim^w(x,y)\,,$$

and for every $\varepsilon > 0$ there exist $x \in E$ and $y \in F$ such that

$$\dim^w(x) \geq \dim_H(E) - \varepsilon \quad \text{and} \quad \dim^{w,x}(y) \geq \dim_H(F) - \varepsilon\,.$$

For this x and y,

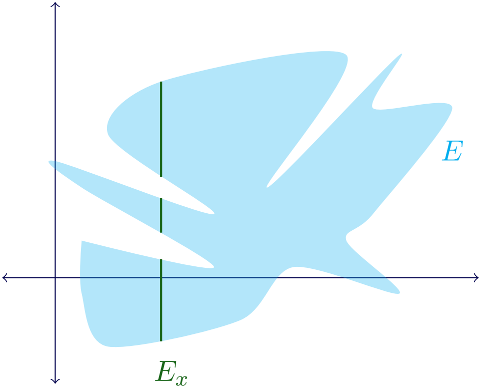
$$\begin{aligned}
\dim_H(E \times F) &\geq \dim^w(x,y) \\
&\geq \dim^w(x) + \dim^w(y|x) \\
&\geq \dim^w(x) + \dim^{w,x}(y) \\
&\geq \dim_H(E) + \dim_H(F) - 2\varepsilon\,.
\end{aligned}$$

Let $\varepsilon \to 0$.

Slicing Theorem (Marstrand 1954)

Let $E \subseteq \mathbb{R}^2$ be a Borel set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1 \,.$$



Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1\,.$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y)\,,$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y)\,,$$

and for all $\varepsilon > 0$ and $x \in \mathbb{R}$, there is a point $(x,y) \in E_x$ such that

$$\dim^{w,x}(x,y) \geq \dim_H(E_x) - \varepsilon\,.$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y)\,,$$

and for all $\varepsilon > 0$ and $x \in \mathbb{R}$, there is a point $(x,y) \in E_x$ such that

$$\dim^{w,x}(x,y) \geq \dim_H(E_x) - \varepsilon\,.$$

Since $(x,y) \in E$, we have

$$\dim_H(E) \geq \dim^w(x,y)$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1 \,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y) \,,$$

and for all $\varepsilon > 0$ and $x \in \mathbb{R}$, there is a point $(x,y) \in E_x$ such that

$$\dim^{w,x}(x,y) \geq \dim_H(E_x) - \varepsilon \,.$$

Since $(x,y) \in E$, we have

$$\begin{aligned} \dim_H(E) &\geq \dim^w(x,y) \\ &\geq \dim^w(x) + \dim^w(y|x) \end{aligned}$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y)\,,$$

and for all $\varepsilon > 0$ and $x \in \mathbb{R}$, there is a point $(x,y) \in E_x$ such that

$$\dim^{w,x}(x,y) \geq \dim_H(E_x) - \varepsilon\,.$$

Since $(x,y) \in E$, we have

$$\begin{aligned}
\dim_H(E) &\geq \dim^w(x,y) \\
&\geq \dim^w(x) + \dim^w(y|x) \\
&\geq \dim^w(x) + \dim^{w,x}(y)
\end{aligned}$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y)\,,$$

and for all $\varepsilon > 0$ and $x \in \mathbb{R}$, there is a point $(x,y) \in E_x$ such that

$$\dim^{w,x}(x,y) \geq \dim_H(E_x) - \varepsilon\,.$$

Since $(x,y) \in E$, we have

$$\begin{aligned}
\dim_H(E) &\geq \dim^w(x,y) \\
&\geq \dim^w(x) + \dim^w(y|x) \\
&\geq \dim^w(x) + \dim^{w,x}(y) \\
&= \dim^w(x) + \dim^{w,x}(x,y)
\end{aligned}$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1\,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y)\,,$$

and for all $\varepsilon > 0$ and $x \in \mathbb{R}$, there is a point $(x,y) \in E_x$ such that

$$\dim^{w,x}(x,y) \geq \dim_H(E_x) - \varepsilon\,.$$

Since $(x,y) \in E$, we have

$$\begin{aligned}
\dim_H(E) &\geq \dim^w(x,y) \\
&\geq \dim^w(x) + \dim^w(y|x) \\
&\geq \dim^w(x) + \dim^{w,x}(y) \\
&= \dim^w(x) + \dim^{w,x}(x,y) \\
&\geq \dim^w(x) + \dim_H(E_x) - \varepsilon\,.
\end{aligned}$$

Slicing Theorem for Arbitrary Sets (N. Lutz '16)

Let $E \subseteq \mathbb{R}^2$ be any set with $\dim_H(E) \geq 1$, and let E_x be the vertical slice of E at x. Then for almost all $x \in \mathbb{R}$,

$$\dim_H(E_x) \leq \dim_H(E) - 1 \,.$$

Proof. By the point-to-set principle, there is an oracle w such that

$$\dim_H(E) = \sup_{(x,y)\in E} \dim^w(x,y) \,,$$

and for all $\varepsilon > 0$ and $x \in \mathbb{R}$, there is a point $(x,y) \in E_x$ such that

$$\dim^{w,x}(x,y) \geq \dim_H(E_x) - \varepsilon \,.$$

Since $(x,y) \in E$, we have

$$\begin{aligned} \dim_H(E) &\geq \dim^w(x,y) \\ &\geq \dim^w(x) + \dim^w(y|x) \\ &\geq \dim^w(x) + \dim^{w,x}(y) \\ &= \dim^w(x) + \dim^{w,x}(x,y) \\ &\geq \dim^w(x) + \dim_H(E_x) - \varepsilon \,. \end{aligned}$$

Recall that $\dim^w(x) = 1$ for almost all $x \in \mathbb{R}$, and let $\varepsilon \to 0$. □

Conclusion

Algorithmic dimension provides a simple, intuitive, and powerful approach to problems in classical fractal geometry.

Conclusion

Algorithmic dimension provides a simple, intuitive, and powerful approach to problems in classical fractal geometry.

- This approach has also been used to bound the dimension of generalized Furstenberg sets (related to Kakeya sets).

Conclusion

Algorithmic dimension provides a simple, intuitive, and powerful approach to problems in classical fractal geometry.

- This approach has also been used to bound the dimension of generalized Furstenberg sets (related to Kakeya sets).
- Although the simple proofs in this work operated at the "higher level" of dimension, that proof is significantly more involved and reasons about Kolmogorov complexity directly.

Conclusion

Algorithmic dimension provides a simple, intuitive, and powerful approach to problems in classical fractal geometry.

- This approach has also been used to bound the dimension of generalized Furstenberg sets (related to Kakeya sets).
- Although the simple proofs in this work operated at the "higher level" of dimension, that proof is significantly more involved and reasons about Kolmogorov complexity directly.
- Objective: Further strengthen the connections between geometric measure theory and algorithmic information theory, i.e., generalize and refine the point-to-set principle.

Conclusion

Algorithmic dimension provides a simple, intuitive, and powerful approach to problems in classical fractal geometry.

- This approach has also been used to bound the dimension of generalized Furstenberg sets (related to Kakeya sets).
- Although the simple proofs in this work operated at the "higher level" of dimension, that proof is significantly more involved and reasons about Kolmogorov complexity directly.
- Objective: Further strengthen the connections between geometric measure theory and algorithmic information theory, i.e., generalize and refine the point-to-set principle.
- Broader project: Systematically re-examine the foundations of fractal geometry through this pointwise lens.