A stratified pointfree definition of probability via constructive natural density

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Kolmogorov's probability became the standard notion
but it is **not informative** about the assignment of probability.

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3. There exists a unique extension of $\mathbb{P}$ to $\sigma(\mathcal{R}) = \mathcal{B}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$.

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5. These are exactly those measurable fuzzy sets for which $f \Rightarrow \perp$ and $\neg f$ coincide.

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We try to develop this idea in a constructive framework

Minimalist Foundation (+ AC!)

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   If (e, γ) and (e', γ') are equal actual events,
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3. $\mathbb{P}(\mathrm{e}, \gamma) := \Phi(\mathrm{e}) \circ \gamma$ is a well-defined operation
   from actual events to Bishop reals.

Properties of probability (1)

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2. (Modularity) If $(\mathsf{e}, \gamma), (\mathsf{e}', \gamma'), (\mathsf{e} \vee \mathsf{e}', \gamma''), (\mathsf{e} \wedge \mathsf{e}', \gamma''') \in \mathcal{A}$, then
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or…

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THANK YOU FOR YOUR ATTENTION