On real numbers in the Minimalist Foundation

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Continuity, Computability, Constructivity-From Logic to Algorithms

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Short Abstract

The variety of definitions of real numbers

as paradigmatic examples

of peculiar characteristics

of the Minimalist Foundation **MF**

Abstract

- Primitive def. of logic on type theory in **MF**

  distinct notions of real numbers:

  regular Cauchy sequences à la Bishop as typed-terms

  regular Cauchy sequences as functional relations

  called Brouwer reals

  Dedekind real numbers

- constructivity of **MF**: it enjoys a realizability model

  where all the above definitions are equivalent

  and all real numbers are computable

- minimality of **MF**

  $\Rightarrow$ **strict predicativity** of **MF**

  Regular Cauchy sequences as functional relations

  and Dedekind reals do not form sets but proper collections

  $\Rightarrow$ we need to work on them via point-free topology

Constructivity of **MF**

as a foundation of constructive mathematics

is expressed by its many-level structure we build a many-level foundation

for constructive mathematics

to make

EXPLICIT

the IMPLICIT computational contents

of constructive mathematics

indeed.....

what is constructive mathematics?

CONSTRUCTIVE mathematics
=
IMPLICIT COMPUTATIONAL mathematics

$\Downarrow$

| with NO explicit use of TURING MACHINES
| BUT with COMPUTATIONS by CONSTRUCTION |

$\Downarrow$

| constructive mathematician is an implicit programmer!! |

[G. Sambin] Doing Without Turing Machines: Constructivism and Formal Topology.
In "Computation and Logic in the Real World". LNCS 4497, 2007

CONSTRUCTIVE proofs

=

SOME programs

What is a constructive foundational theory?

a foundational theory is **constructive** = its proofs have a **computational interpretation**

i.e. there exists a computable model, called *realizability* model,

where

we can compute witnesses
of proven existential statements
even under hypothesis Γ

i.e. in the *realizability* model

$\exists x \varepsilon A\, \phi(x)$ **true** under hypothesis Γ

$\Downarrow$

there exists a PROGRAM calculating c_Γ depending on Γ

s.t. $\phi(c_\Gamma)$ **true** under hypothesis Γ

$\Rightarrow$ in the realizability model

- **the choice rule** (**CR**)

$\exists x \varepsilon A\ \phi(x)$ **true**	under hypothesis Γ
$\Downarrow$	
there exists a function calculating $f(x)$ such that	
$\phi(f(x))$ **true**	under hypothesis $x \in \Gamma$

- "all functions of the models are computable"

must be valid!

in [M. Sambin-2005] we required

realizability model validates AC+ CT

i.e. previous requirements hold internally

$$(AC) \qquad \forall x \in A \; \exists y \in B \; R(x,y) \quad \longrightarrow \quad \exists f \in A \rightarrow B \;\; \forall x \in A \;\; R(\, x \,,\, f(x)\,)$$

$$(CT) \qquad \forall f \in \mathsf{N}at \rightarrow \mathsf{N}at \qquad \exists e \in \mathsf{N}at$$
$$(\, \forall x \in \mathsf{N}at \;\; \exists y \in \mathsf{N}at \quad T(e,x,y) \;\, \& \;\, U(y) =_{\mathsf{N}at} f(x)\,)$$

to view COMPUTATIONAL CONTENTS of constructive mathematics

jointly with G. Sambin

FORMALIZE constructive mathematics

in TWO-LEVEL foundation

conciliating TWO different languages:

abstract mathematics	in usual set-theoretic language
computational mathematics	in a programming language to view proofs-as-programs

but this is not enough...

need of a INTERACTIVE THEOREM PROVER...

better to use an INTERACTIVE THEOREM PROVER

to develop COMPUTER-AIDED FORMALIZED PROOFS

+ PROGRAM extraction

hopefully in intensional type theory

What **foundation** *for COMPUTER-AIDED formalization of proofs?*

(j.w.w. G. Sambin)

a FORMAL Constructive Foundation should include

extensional LANGUAGE of abstract maths

as usual set theoretic language

↓ interpreted in

intensional trustable base

for an INTERACTIVE prover

↓ interpreted in

a PROGRAMMING LANGUAGE

acting as a realizability model

(for proofs-as-programs extraction)

our notion of constructive foundation

= a two-level foundation + a realizability level

PURE Foundation	extensional level (used by mathematicians to do their proofs)
	$\Downarrow$ interpreted via a **QUOTIENT model**
	intensional level (language of computer-aided formalized proofs)

$\Downarrow$

realizability level (used by computer scientists to extract programs)

in our notion of constructive foundation

the realizability model

where to **extract**

programs from constructive proofs

is *NOT part* of the PURE foundational structure

but only *a PROPERTY* of the intensional level

why is the realizability level not part of the Pure Foundation?

for example

the statement "all functions are COMPUTABLE"

may hold INTERNALLY at the realizability level

(for ex. in Kleene realizability of HA)

BUT it is NOT compatible

with CLASSICAL extensional foundations

in our notion of Constructive Foundation we combine different languages

language of *(local) AXIOMATIC SET THEORY*	for extensional level
language of *CATEGORY THEORY*	algebraic structure to link intensional/extensional levels via a quotient completion
language of TYPE THEORY	for intensional level
computational language	for realizability level

need to use CATEGORY THEORY

to express the link between extensional/intensional levels:

use
notion of ELEMENTARY QUOTIENT COMPLETION/EXACT completion
(in the language of CATEGORY THEORY)
relative to a suitable **Lawvere's doctrine**

in:

[M.E.M.-Rosolini'13] "Quotient completion for the foundation of constructive mathematics", Logica Universalis

[M.E.M.-Rosolini'13] "Elementary quotient completion", Theory and Applications of Categories

[M.E.M.-Rosolini'15] "Unifying exact completions", Applied Categorical Structures

what examples of pure TWO-level FOUNDATIONS?

our TWO-LEVEL Minimalist Foundation called **MF**

ideated in [Maietti-Sambin'05] and completed in [Maietti'09]

both levels of **MF** are based on **DEPENDENT TYPE THEORIES** à la Martin-Löf with primitive def. of logic

the pure TWO-LEVEL structure of the Minimalist Foundation

from [Maietti'09]

- its intensional level

  = a PREDICATIVE VERSION of the Calculus of Inductive Constructions

- its extensional level

  is a PREDICATIVE LOCAL set theory

  (NO choice principles)

  a predicative version of the internal theory of *elementary toposes*

  (it has power-collections of sets)

What realizability level for **MF** *?*

Martin-Löf's type theory

or

an extension of Kleene realizability

of **intensional level of MF**+ Axiom of Choice + Formal Church's thesis

as in

H. Ishihara, M.E.M., S. Maschio, T. Streicher

Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice

Why **MF** *is called minimalist?*

because **MF** is a common core

among most relevant constructive foundations

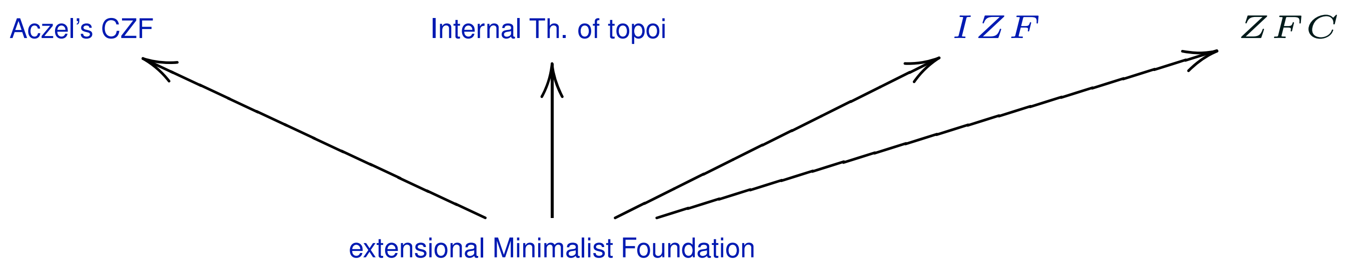
Plurality of constructive foundations $\Rightarrow$ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory Feferman's constructive expl. maths

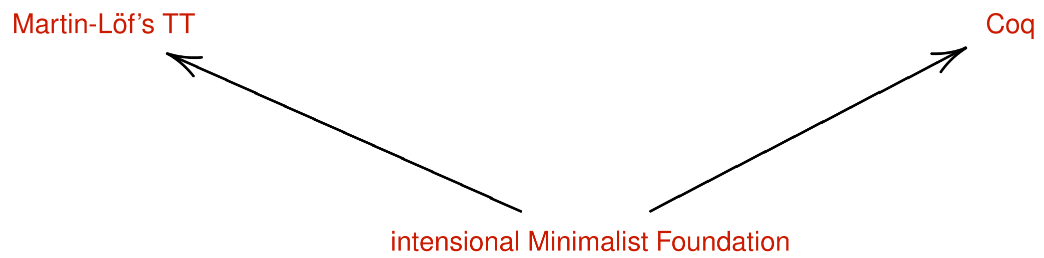
the **MINIMALIST FOUNDATION** is a common core

WARNING on compatibility

relate extensional theories with the extensional level of **MF**



relate intensional theories with the intensional level of **MF**



in **MF**: *two notions of functions*

NO choice principles are valid in **MF**

as in the type theory of the proof-assistant COQ

$$\Downarrow$$

for A, B sets

1. function as a functional relation,

i.e. a (small) proposition $R(x, y)$ s.t.

$$\forall x \in A \, \exists! y \in B \; R(x, y)$$

2.functions as a (Bishop's) operation(= or typed theoretic function)

$$\lambda x.f(x) \in \Pi_{x \in A} B = OP(A, B)$$

type-theoretic functions are defined primitively!!!

$$Graph(-) : Op(A, B) \rightarrow Fun(A, B)$$

`proper embedding`

as usual in constructive mathematics

in **MF** we have

various notions of real numbers.

NO choice principles *in* **MF**

$$\Downarrow$$

distinct notions of real numbers:

regular Cauchy sequences à la Bishop as typed-terms called Bishop reals
$\neq$ (NO axiom of unique choice in **MF**)
regular Cauchy sequences as functional relations called Brouwer reals
$\neq$ (NO countable choice in **MF**)
Dedekind cuts (lower + upper) called Dedekind reals

Dedekind real numbers in **MF**

as in [Fourman-Hyland'79]

A Dedekind real number is a Dedekind cut

$$(L, U)$$

with $L, U \subseteq \mathbb{Q}$ non empty and:

(disjointness)	$\forall q \in \mathbb{Q} \ \neg(\, q \,\epsilon\, U \,\&\, q \,\epsilon\, L \,)$
(L-openess)	$\forall p \,\epsilon\, L \ \exists q \,\epsilon\, L \ \ p < q$
(U-openess)	$\forall q \,\epsilon\, U \ \exists p \,\epsilon\, U \ \ p < q$
(L-monotonicity)	$\forall q \,\epsilon\, L \ \forall p \in \mathbb{Q} \ (\, p < q \ \rightarrow \ p \,\epsilon\, L \,)$
(U-monotonicity)	$\forall p \,\epsilon\, U \ \forall q \in \mathbb{Q} \ (\, p < q \ \rightarrow \ q \,\epsilon\, U \,)$
(locatedness)	$\forall q \in \mathbb{Q} \, \forall p \in \mathbb{Q} \ (\, p < q \ \rightarrow \ p \,\epsilon\, L \ \vee \ q \,\epsilon\, U \,)$

Bishop reals in **MF**

Bishop reals $\equiv$ quotient of regular Cauchy sequences

under Cauchy condition

A Bishop real is a regular rational sequence $x_n \in \mathbb{Q}\ [n \in Nat^+]$
given by a **typed term** in **MF**
such that for $n, m \in \mathrm{N}at^+$

$$|\ x_n - x_m\ | \leq\ 1/n + 1/m$$

two Bishop real numbers

$$x_n \in \mathbb{Q}\,[n \in Nat^+] \qquad\qquad y_n \in \mathbb{Q}\,[n \in Nat^+]$$

are equal

iff

for $n, m \in$ Nat^+

$$\mid x_n - y_n \mid \leq \; 2/n$$

Brouwer reals in **MF**

Brouwer reals $\equiv$ regular Cauchy sequences as functional relations

i.e. rational sequences given by functional relations

$$R(n,x)\ prop_s \quad [n \in Nat^+, x \in \mathbb{Q}]$$

such that

$$\forall\, p \in \mathbb{Q}\, \forall\, q \in \mathbb{Q}\ (\ R(n,p)\ \&\ R(m,q)\ \rightarrow |\ q - p\ | \leq\ 1/n + 1/m\ )$$

From primitive logic + type theory in **MF**

$$\Downarrow$$

Bishop reals $\longrightarrow$ Brouwer reals $\longrightarrow$ Dedekind reals

all proper embeddings

as in the type theory of the proof-assistant Coq all the definitions of real numbers in **MF**

are equivalent

in the extension of Kleene realizability model

and they are all computable!!

in the extension of Kleene realizability interpretation to **MF**

because of validity of

Axiom of Choice + Formal Church's thesis

in the interpretation of the intensional level of **MF**

in

H. Ishihara, M.E.M., S. Maschio, T. Streicher

Consistency of the Minimalist Foundation with Church's thesis and Axiom of Choice

$\Downarrow$

in the lifted interpretation of the extensional level of **MF**
Dedekind reals = Brouwer reals= Bishop reals
and they are all computable!!!

How to get a model of **MF**

Any model of the intensional level of **MF**

can be turned into a model of the extensional level of **MF**

via a **elementary QUOTIENT COMPLETION**

as in

[M.-Rosolini'13] "Quotient completion for the foundation of constructive mathematics", Logica Universalis.

[M.-Rosolini'13] ""Elementary quotient completion", Theory and applications of categories.

A **key novelty** *of* **MF**

MF is strictly predicative

à la Feferman

$\Downarrow$

our proposal:

MF = base for constructive **reverse mathematics**

open problem:

find the proof-theoretic strength of **MF**

(hopefully that of Heyting arithmetics!)

from strictly predicativity of **MF**

CONTRARY to the type theory in the proof-assistant COQ

for A, B **MF**-sets:

Functional relations from A to B do NOT always form a set
=Exponentiation $Fun(A, B)$ of functional relations is not always a set
$\neq$
Operations (typed-theoretic terms) from A to B do form a set
= Exponentiation $Op(A, B)$ is a set

in **MF**

exponentiation of functional relations is NOT always a set

$\Downarrow$

power-collections of not empty sets are NOT generally sets

even when classical logic is added

$\Downarrow$

MF is compatible with **classical predicativity**

Aczel's CZF NOT compatible with classical predicativity

Aczel's Constructive Zermelo-Fraenkel set theory+ classical logic = IMPREDICATIVE Zermelo Fraenkel theory
it is not predicative in the proof-theoretic strength à la Feferman $\Longrightarrow$ it is NOT minimalist

$\Downarrow$

Aczel's CZF

is NOT compatible with

classical predicative theories à la Feferman

From strict predicativity of **MF**

set of		collection of		collection of
Bishop reals	$\longrightarrow$	Brouwer reals	$\longrightarrow$	Dedekind reals

all proper embeddings

why Brouwer/Dedekind reals do NOT form a set

via a model of the INTENSIONAL LEVEL of **MF**

in the full subcategory $Ass(Eff)$ of ASSEMBLIES

of Hyland's Effective topos:

MF sets	assemblies (X, ϕ) with X countable
operations between sets	as assemblies morphisms
propositions	strong monomorphisms of assemblies
proper collections (= NO sets)	assemblies (X, ϕ) with X not countable

$$\Downarrow$$

NON validity of axiom of unique choice between natural numbers

Brouwer reals and Dedekind reals of **MF**

are interpreted as NOT countable assemblies!

$\Longrightarrow$ they are not **MF**-sets

while Bishop reals are interpreted as computable ones

Axiom of unique choice

$$\forall x \in A \, \exists! y \in B \; R(x, y) \quad \longrightarrow \quad \exists f \in A \to B \, \forall x \in A \; R(x, f(x))$$

turns a functional relation into a type-theoretic function.

$\Rightarrow$ identifies the two distinct notions...

Key properties of assemblies in Eff

well known:

The full subcategory of assemblies Ass(Eff) in Hyland's Effective Topos is a **boolean quasi-topos** with a natural numbers object

seen as a consequence of

j.w.w Fabio Pasquali and Giuseppe Rosolini

The full subcategory of assemblies Ass(Eff) in Hyland's Effective Topos is an **elementary quotient completion** of the elementary doctrine of strong monomorphisms restricted to partitioned assemblies

From strict predicativity of **MF**

set of		collection of		collection of
Bishop reals	$\longrightarrow$	Brouwer reals	$\longrightarrow$	Dedekind reals

all proper embeddings

$$\Downarrow$$

constructive topology in **MF**

(in particular on real numbers)

must be point-free

Topology on real numbers in **MF**

via Martin-Löf and Sambin's notion of formal topology

by using **inductive methods**

topology on Dedekind reals

=

Joyal's formal topology $\mathcal{R}_d$

(classically the right topology!) inductively generated with

ideal points= Dedekind reals

---

topology on Bishop reals

=

"pointwise" topology

as a concrete space by Sambin

with Joyal's formal topology $\mathcal{R}_d$

how to reason on Brouwer real numbers topologically?

future work:

by using Sambin's Positive Topology together with

REPRESENTATIVES of Brouwer reals	=	ideal points of Baire formal topology $\mathcal{BS}$
		on finite regular rational sequence
		via inductive generation of open subsets

+

the formal topology morphism

$$i : \mathcal{BS} \to \mathcal{R}_d$$

from the formal topology of representatives of Brouwer reals as ideal points

to Joyal's topology of Dedekind cuts

Future work

- use **MF** to perform constructive reverse mathematics
  in particular for Bishop constructive analysis
  (w.r.t. use of Bar Induction, Fan theorem)

- extend realizability models to include inductively generated formal topologies
  for extraction of programs from proofs

- **build** a **Minimalist Proof assistant**
  based on three levels of **MF**
  for proof formalization

point-free topology

Martin-Löf-Sambin's formal topology = an approach to predicative point-free topology

formal topology employs $(S, \lhd, \mathsf{Pos})$

S= a set of basic opens

$a \lhd U$ = a cover relation: says when a basic open a is covered by the union of opens in $U \subseteq S$

FORMAL (or IDEAL) POINT= (suitable) completely prime filter

PREDICATIVE constructive POINT-FREE TOPOLOGY helps to describe the hopefully finitary (or inductive) structure of a topological space

whose points can be ONLY described in infinitary way!!

(and they do not form a set)

pointfree presentation of Dedekind reals

Joyal's formal topology $\mathcal{R}_d \equiv (\mathbb{Q} \times \mathbb{Q}, \lhd_{\mathcal{R}}, \mathsf{Pos}_{\mathcal{R}})$

Basic opens are pairs $\langle p, q \rangle$ of rational numbers

whose cover $\lhd_{\mathcal{R}}$ is inductively generated as follows:

$$\frac{q \leq p}{\langle p, q \rangle \lhd_{\mathcal{R}} U} \qquad \frac{\langle p, q \rangle \in U}{\langle p, q \rangle \lhd_{\mathcal{R}} U} \qquad \frac{p' \leq p < q \leq q' \qquad \langle p', q' \rangle \lhd_{\mathcal{R}} U}{\langle p, q \rangle \lhd_{\mathcal{R}} U}$$

$$\frac{p \leq r < s \leq q \qquad \langle p, s \rangle \lhd_{\mathcal{R}} U \quad \langle r, q \rangle \lhd_{\mathcal{R}} U}{\langle p, q \rangle \lhd_{\mathcal{R}} U} \qquad \mathsf{wc}\ \frac{wc(\langle p, q \rangle) \lhd_{\mathcal{R}} U}{\langle p, q \rangle \lhd_{\mathcal{R}} U}$$

where

$$wc(\langle p, q \rangle) \equiv \{ \langle p', q' \rangle \in \mathbb{Q} \times \mathbb{Q} \mid p < p' < q' < q \}$$

representatives of Brouwer reals as ideal points

representative of Brouwer reals= ideal points of the point-free topology

$$\mathcal{BS} \equiv (BS, \lhd_{\mathcal{BS}}, \mathsf{Pos}_{\mathcal{BS}})$$

where BS = set of finite regular sequences

set of $l \in List(\mathbb{Q})$ such that

$$\forall n \in \mathsf{N}at^{+} \, \forall m \in \mathsf{N}at^{+} \, (\, n \leq \mathsf{lh}(l) \, \& \, m \leq \mathsf{lh}(l)$$
$$\rightarrow \mid l_m - l_n \mid \leq 1/n + 1/m \,)$$

and $\lhd_{\mathcal{BS}}$ is the formal topology of sequences

(as Baire topology) restricted to finite regular sequences!!

$l \lhd_{\mathcal{BS}} U$ means

" any sequence passing through l passes through an element $u \,\epsilon\, U$"

$l \lhd_{\mathcal{BS}} U$ is inductively generated by the following rules

$$\text{rfl}\ \frac{l \,\epsilon\, V}{l \lhd_{\mathcal{C}} V} \qquad \leq \frac{s \sqsubseteq l \qquad l \lhd_{\mathcal{C}} V}{s \lhd_{\mathcal{C}} V} \qquad \text{tr}\ \frac{[l * \mathbb{Q}]_b \lhd_{\mathcal{C}} V}{l \lhd_{\mathcal{C}} V}$$

where $s \sqsubseteq l$ = l is an initial segment of s

Point-free embedding

The proof that a representative of a Brouwer real defines a Dedekind real

becomes the proof that

$$i : \mathcal{BS} \to \mathcal{R}_d$$

is a formal topology morphism

from the formal topology $\mathcal{BS}$ with representatives of Brouwer reals as ideal points

to Joyal's topology $\mathcal{R}_d$ of Dedekind cuts

where for $l \in \mathbb{Q}^*$ and $p, q \in \mathbb{Q}$ as the relation

$$l \; i_b \; < p, q >$$

$$\equiv$$

$$\exists\, n \in \mathsf{N}at^+ \;\; n \leq \mathsf{lh}(l) \;\&$$

$$p \; < \; l_n - 2/n \; < \; l_n + 2/n \; < \; q$$

moral: we do calculations ONLY on FINITE APPROXIMATIONS of both kinds of reals

Many-levels constructive foundation

to be implemented in a **Minimalist proof assistant**