The Computational Content of the Constructive Kruskal Tree Theorem

Dominique Larchey-Wendling

TYPES team

LORIA – CNRS

Nancy, France

http://www.loria.fr/~larchey/Kruskal

Continuity, Computability, Constructivity -

From Logic to Algorithms, CCC 2017

Well Quasi Orders (WQO) 1/2

- Important concept in Computer Science:
  - strenghtens well-foundedness, more stable
  - termination of rewriting (Dershowitz, RPO)
  - size-change termination, terminator (Vytiniostis, Coquand ...)
- Important concept in Mathematics:
  - Dickson's lemma, Higman's lemma
  - Higman's theorem, Kruskal's tree theorem
  - Robertson-Seymour theorem (graph minor theorem)
  - Unprovability result: Kruskal theorem not in PA (Friedman)

Well Quasi Orders (WQO) 2/2

- for $\leq$ a quasi order over X: reflexive & transitive binary relation
- several **classically** equivalent definitions (see e.g. JGL 2013)
  - almost full: each $(x_i)_{i\in\mathbb{N}}$ has a *good pair* ($x_i \leq x_j$ with $i < j$)
  - $\leq$ well-founded and no ∞ antichain
  - finite basis: $U = {\uparrow}U$ implies $U = {\uparrow}F$ for some finite F
  - $\{{\downarrow}U \mid U \subseteq X\}$ well-founded by $\subset$
- many of these equivalences **do not hold** intuitionistically

WQOs are stable under type constructs

- Given a WQO $\leq$ on X, we can lift $\leq$ to WQOs on:

  Higman lemma: `list`(X) with `subword`$(\leq)$

  Higman thm: `btree`(k, X) with `emb_product`$(\leq)$ (any $k \in \mathbb{N}$)

  Kruskal theorem: `tree`(X) with `emb_homeo`$(\leq)$

- These theorem are *closure properties* of the class of WQOs

- Other noticable results:

  Dickson's lemma: $(\mathbb{N}^k, \leq)$ is a WQO

  Finite sequence thm: `list`$(\mathbb{N})$ WQO under `subword`$(\leq)$

  Ramsey theorem: $\leq_1$ and $\leq_2$ WQOs imply $\leq_1 \times \leq_2$ WQO

What Intuitionistic Kruskal Tree Theorem?

- The meaning of those closure theorems intuitionistically:
  - depends of what is a WQO (which definition?)
  - but not on e.g. `emb_homeo` which has an inductive definition
- What is a suitable intuitionistic definition of WQO ?
  - quasi-order does not play an important/difficult role
  - should be classically equivalent to the usual definition
  - should intuitionistically imply almost full
  - intuitionistic WQOs must be stable under liftings
- Allow the proof and use of Ramsey, Higman, Kruskal... results

Intuitionistic formulations of WQOs 1/2

- Almost full relations (Veldman&Bezem 93)
  - each $(x_i)_{i \in \mathbb{N}}$ has $x_i \, R \, x_j$ with $i < j$
  - works for Higman and Kruskal theorems (Veldman 04)
  - uses *stumps* over $\mathbb{N}$ which require *Brouwer's thesis*
- Bar induction (Coquand&Fridlender 93)
  - $\texttt{bar extends } (\texttt{good}\, R)\ [\,]$
  - works for the general Higman lemma (Fridlender 97)
- Well-foundedness (Seisenberger 2003)
  - $\texttt{extends}^{(-1)}$ is well-founded on $\texttt{Bad}(R)$
  - works for Higman lemma and Kruskal theorem
  - requires *decidability* of R

Intuitionistic formulations of WQOs 2/2

- Almost full relations (Vytiniostis&Coquand&Wahlstedt 12)
  - `af`(R) inductively defined
  - works for Ramsey theorem
  - intuitionistically equivalent to `bar extends` (`good` R) [ ]
- Seisenberger's definition not equiv. to Coquand&Fridlender for undecidable R
- Veldman&Bezem definition works for R over $\mathbb{N}$ (not over arbitrary types) but requires Brouwer's thesis
- Let us introduce
  - bar inductive predicates
  - Coquand et al. inductive definition of almost full

Bar inductive predicate, accessibility predicate

- Given $\mathcal{T} : X \to X \to \texttt{Prop}$, $x : X$ and $Q : X \to \texttt{Prop}$
- Q *bars* x if every ∞ $\mathcal{T}$-path from x meets Q
- x *is accessible* if every ∞ $\mathcal{T}$-path from x meets $\_ \mapsto \texttt{False}$
- Inductive definitions ($\texttt{Prop}$ or $\texttt{Type}$) are stronger (intui.)

$$\frac{Q\ x}{\texttt{bar}\ \mathcal{T}\ Q\ x} \qquad \frac{\forall y, \mathcal{T}\ x\ y \to \texttt{bar}\ \mathcal{T}\ Q\ y}{\texttt{bar}\ \mathcal{T}\ Q\ x} \quad \Bigg| \quad \frac{\forall y, \mathcal{T}\ x\ y \to \texttt{acc}\ \mathcal{T}\ y}{\texttt{acc}\ \mathcal{T}\ x}$$

- Axioms (like Brouwer's bar thesis) for equivalence
- Obviously: $\texttt{acc}\ \mathcal{T}\ x$ iff $\texttt{bar}\ \mathcal{T}\ (\_ \mapsto \texttt{False})\ x$

Bar inductive predicate and the FAN theorem

- inductive FAN theorem: $\boxed{\texttt{bar}\ \mathcal{T}\ Q\ x \to \texttt{bar}\ \mathcal{T}^\circ\ \forall Q\ [x]}$
  - for $\texttt{bar}\ \mathcal{T} : (X \to \texttt{Prop}) \to (X \to \texttt{Prop})$
  - and *monotonic* Q: $\forall x\, y, \mathcal{T}\ x\ y \to Q\ x \to Q\ y$
  - $\mathcal{T}^\circ\ l\ m$ iff $\forall y, y \in m \to \exists x, x \in l \land \mathcal{T}\ x\ y$ (direct image)
  - $(\forall Q)\ l$ iff $\forall x, x \in l \to Q\ x$ (finite quantification)
- for $\texttt{bar}_t\ \mathcal{T} : (X \to \texttt{Prop}) \to (X \to \texttt{Type})$
  - FAN is not provable in this informative case
  - the relation $\mathcal{T}^\circ$ hides the relation between y and x
  - possible solution: restrict $\mathcal{T}^\circ$

Bar inductive predicate and list extensions

- We use $\texttt{bar}\ \mathcal{T}\ Q$ with $\mathcal{T} = \texttt{extends}$ (and $Q = \texttt{good}\ R$)
  - $\texttt{extends}\ l\ m$ iff $m = \_ :: l$
  - $\texttt{good}\ R\ ll$ iff $ll = l \mathbin{++} \boxed{b} :: m \mathbin{++} \boxed{a} :: r$ for some $a\ R\ b$
  - $\texttt{good}$ has an easy inductive definition, beware of snoc lists
  - $\texttt{bar}^l = \texttt{bar extends} : (\texttt{list}\,X \to \texttt{Prop}) \to (\texttt{list}\,X \to \texttt{Prop})$

$$\frac{Q\ l}{\texttt{bar}^l\ Q\ l} \qquad \frac{\forall x, \texttt{bar}^l\ Q\ (x :: l)}{\texttt{bar}^l\ Q\ l}$$

- $\texttt{bar}^l\ (\texttt{good}\ R)\ [\,]$ iff $\boxed{\text{iterated exts of } [\,] \text{ meets a good list}}$
- every infinite sequence contains a good pair (almost full)

The Informative FAN theorem (Fridlender)

- $\texttt{bar}_t^l = \texttt{bar}_t \ \texttt{extends} : (\texttt{list}\, X \to \texttt{Prop}) \to (\texttt{list}\, X \to \texttt{Type})$

$$\frac{Q\ l}{\texttt{bar}_t^l\ Q\ l} \qquad \frac{\forall x, \texttt{bar}_t^l\ Q\ (x :: l)}{\texttt{bar}_t^l\ Q\ l}$$

- the list of choice sequences:

$$[x_1; \ldots; x_n] \in \texttt{list\_expo}\ [l_1; \ldots; l_n] \iff x_1 \in l_1 \wedge \cdots \wedge x_n \in l_n$$

- an informative instance of the FAN theorem (Q monotonic):

$$\boxed{\texttt{bar}_t^l\ Q\ [\,] \to \texttt{bar}_t^l\ (\forall Q \circ \texttt{list\_expo})\ [\,]}$$

- Q is met $\boxed{\text{uniformly}}$ among choices sequences

Inductive bars of decidable predicates

- $\texttt{bar}_t$ is obviously stronger than $\texttt{bar}$
- but $\texttt{bar}\ \mathcal{T}\ Q\ l$ not enough to build $\texttt{bar}_t\ \mathcal{T}\ Q\ l$
- however, it is sufficient when $\boxed{Q \text{ is decidable}}$

$$(\forall l, \{Q\ l\} + \{\neg Q\ l\}) \to \forall l, \texttt{bar}\ \mathcal{T}\ Q\ l \to \texttt{bar}_t\ \mathcal{T}\ Q\ l$$

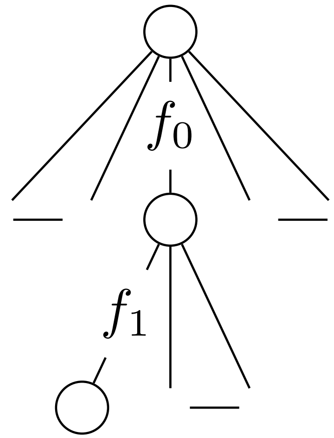
- if Q has a decision term then missing info. can be reconstructed
- also, $\texttt{bar}_t\ Q\ x$ is equivalent to $\texttt{acc}\ (u\ v \mapsto \mathcal{T}\ u\ v \wedge \neg Q\ u)\ x$
- $\texttt{bar}_t^l\ (\texttt{good}\, R)$ and $\texttt{bar}_t^l\ (\texttt{good}\, R)$ equivalent when R decidable

Well-founded trees over a type X

- Well-founded trees $\texttt{wft}(X)$, lfp of $\boxed{\texttt{wft}(X) = \{\star\} + X \to \texttt{wft}(X)}$

$$\frac{\star : \texttt{unit}}{\texttt{inl}\,\star : \texttt{wft}(X)} \qquad \frac{g : X \to \texttt{wft}(X)}{\texttt{inr}\,g : \texttt{wft}(X)}$$

- Given a branch $f : \mathbb{N} \to X$, compute its height:

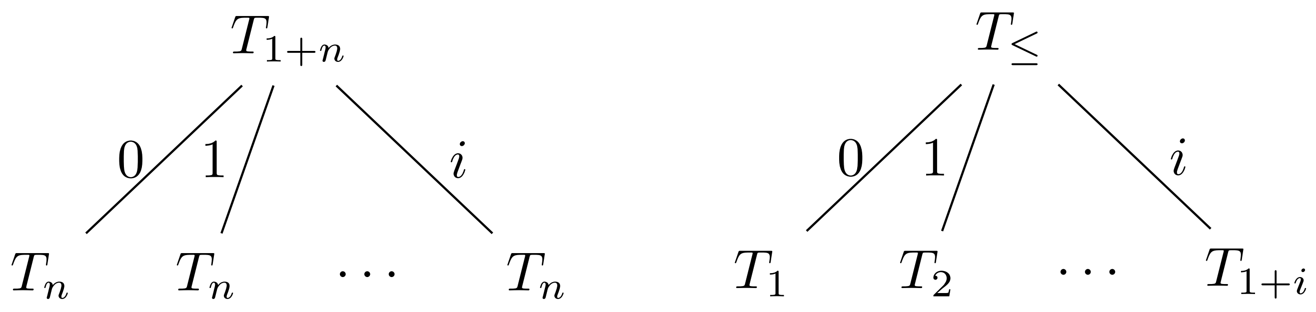


- $f(1+\cdot) = x \mapsto f(1+x)$
- $\texttt{ht}(\texttt{inl}\,\star, \_) = 0$
- $\texttt{ht}(\texttt{inr}\,g, f) = 1 + \texttt{ht}(g(f_0), f(1+\cdot))$

- $\texttt{wft}(X)$ collects *bounds* for any sequence $f : \mathbb{N} \to X$
- Veldman's stumps are sets of branches of trees in $\texttt{wft}(\mathbb{N})$

A well-founded tree for $(\mathbb{N}, \leq)$

- Property: $\forall f : \mathbb{N} \to \mathbb{N},\ \exists i < j < 2 + f_0,\ f_i \leq f_j$
- In $\mathtt{wft}(\mathbb{N})$, we define T_n the tree of uniform height n:
  - $T_0 = \mathtt{inl}(\star)$ and $T_{1+n} = \mathtt{inr}(\_ \mapsto T_n)$
  - for any $f : \mathbb{N} \to \mathbb{N}$, $\mathtt{ht}(T_n, f) = n$
- And $T_{\leq} = \mathtt{inr}(n \mapsto T_{1+n})$



- Hence $\mathtt{ht}(T_{\leq}, f) = 1 + \mathtt{ht}(T_{1+f_0}, f(1 + \cdot)) = 2 + f_0$

Computational content of inductive bar predicates

- recall $\texttt{wft}(X) : \texttt{Type}$ inductivelly defined by

$$\frac{\star : \texttt{unit}}{\texttt{inl}\,\star : \texttt{wft}(X)} \qquad \frac{g : X \to \texttt{wft}(X)}{\texttt{inr}\,g : \texttt{wft}(X)}$$

- $\texttt{bar\_securedby}\ Q : \texttt{wft}(X) \to \texttt{list}\,X \to \texttt{Prop}$
  - $\texttt{bar\_securedby}\ Q\ (\texttt{inl}\,\star)\ l = Q\ l$
  - $\texttt{bar\_securedby}\ Q\ (\texttt{inr}\,g)\ l = \forall x, \texttt{bar\_securedby}\ Q\ (g\,x)\ (x :: l)$
- $\texttt{bar}_t^l\ Q\ l \iff \{t : \texttt{wft}(X) \mid \texttt{bar\_securedby}\ Q\ t\ l\}$
- $t : \texttt{wft}(X)$ is the computational content of the $\texttt{bar}_t^l$ predicate

Coquand's Almost full relations, step by step

1. Veldman et al.: $\forall f : \mathbb{N} \to X,\ \exists i < j,\ f_i \mathrel{R} f_j$
2. Logically eq. variant: $\forall f : \mathbb{N} \to X,\ \exists n,\ \exists i < j < n,\ f_i \mathrel{R} f_j$
3. Partially informative: $\forall f : \mathbb{N} \to X,\ \big\{n \,\big|\, \exists i < j < n,\ f_i \mathrel{R} f_j\big\}$
4. Variant: $\big\{h : (\mathbb{N} \to X) \to \mathbb{N} \,\big|\, \forall f,\ \exists i < j < h(f),\ f_i \mathrel{R} f_j\big\}$
5. Variant: $\big\{t : \mathtt{wft}(X) \,\big|\, \forall f,\ \exists i < j < \mathtt{ht}(t, f),\ f_i \mathrel{R} f_j\big\}$
6. Coquand et al.: is defined as an inductive predicate $\mathtt{af}_t(R)$

- the prefix of length $\mathtt{ht}(t, f)$ of $f : \mathbb{N} \to X$ contains a good pair
- the *computational content* is (for every sequence $f : \mathbb{N} \to X$):
  - a bound on the size of the search space for good pairs
  - and it is not a good pair

Almost full relations, inductively

- For $X : \texttt{Type}$ and $R : X \to X \to \texttt{Prop}$
- Lifted relation: $x\ (R\!\uparrow\! u)\ y = x\ R\ y \vee u\ R\ x$
  - in $R\!\uparrow\! u$, elements above u are forbidden in bad sequences
- $\texttt{full} : \texttt{rel}_2\, X \to \boxed{\texttt{Prop}}$ and $\texttt{af}_t : \texttt{rel}_2\, X \to \boxed{\texttt{Type}}$

$$\frac{\forall x, y,\ x\ R\ y}{\texttt{full}\, R} \qquad \Bigg| \qquad \frac{\texttt{full}\, R}{\texttt{af}_t\, R} \qquad \frac{\forall u, \texttt{af}_t(R\!\uparrow\! u)}{\texttt{af}_t\, R}$$

- $\texttt{af\_securedby} : \texttt{wft}(X) \to \texttt{rel}_2\, X \to \texttt{Prop}$:
  - $\texttt{af\_securedby}(\texttt{inl}\, \star, R) = \texttt{full}\, R$
  - $\texttt{af\_securedby}(\texttt{inr}\, g, R) = \forall u, \texttt{af\_securedby}(g\ u, R\!\uparrow\! u)$

Almost full relations, equivalent characterizations

- these are intuitionistically "equivalent" (hold in `Type`, not `Prop`):
  - $\texttt{af}_t\, R$
  - $\{t : \texttt{wft}(X) \mid \texttt{af\_securedby}(t, R)\}$
  - $\{t : \texttt{wft}(X) \mid \forall f,\ \exists i < j < \texttt{ht}(t, f),\ f_i\ R\ f_j\}$
  - $\texttt{bar}^l_t\ (\texttt{good}\ R)\ []$
  - $\{t : \texttt{wft}(X) \mid \texttt{bar\_securedby}\ (\texttt{good}\ R)\ t\ []\}$
  - $\{t : \texttt{wft}(X) \mid \forall f, \texttt{good}\ R\ [f_{n-1}; \ldots ; f_0]\}$ where $n = \texttt{ht}(t, f)$
- the tree $t : \texttt{wft}(X)$ might be modified
- to establish $\texttt{af}_t\, R$ iff $\texttt{bar}^l_t\, (\texttt{good}\, R)\, []$, we prove

$$\texttt{af}_t(R \uparrow a_n \uparrow \ldots \uparrow a_1) \ \text{ iff } \ \texttt{bar}^l_t\ (\texttt{good}\, R)\ [a_1, \ldots, a_n]$$

Almost full relations, some properties

- `af_t_refl`: if $\texttt{af}_t\, R$ then $=_X \subseteq R$ (iff in case X is finite)
- `af_t_inc`: if $R \subseteq S$ and $\texttt{af}_t\, R$ then $\texttt{af}_t\, S$
- `af_t_surjective` (easy but very useful):
  - for $f : X \to Y \to \texttt{Prop}$, $R : \texttt{rel}_2\, X$ and $S : \texttt{rel}_2\, Y$
  - if f surjective: $\forall y, \{x \mid f\ x\ y\}$
  - if f morphism: $f\ x_1\ y_1$ and $f\ x_2\ y_2$ and $x_1\ R\ x_2$ imply $y_1\ S\ y_2$
  - then $\texttt{af}_t\, R$ implies $\texttt{af}_t\, S$
- Ramsey (Coquand): $\texttt{af}_t\, R$ and $\texttt{af}_t\, S$ imply $\texttt{af}_t(R \cap S)$
  - he deduces $\texttt{af}_t(R \times S)$ and $\texttt{af}_t(R + S)$

The Intuitionistic Ramsey Theorem (Coquand)

- By induction on the arity: $\texttt{af}_t\, R$ and $\texttt{af}_t\, S$ imply $\texttt{af}_t(R \cap S)$
- Curry-isomorphically: $\texttt{af}_t\, R$ and $\texttt{af}_t\, S$ imply $\texttt{af}_t(R \times S)$
- Dickson's lemma: $\texttt{af}_t\,(\leq_{\mathbb{N}} \times \cdots \times \leq_{\mathbb{N}})$
- Classical Ramsey (*not provable* intuitionistically):
  - for $X : \texttt{Type}$ infinite and $R : X \to X \to \texttt{Prop}$ define
  - $R_0\ n\ m \iff n = m \vee \neg R\ n\ m$
  - $R_1\ n\ m \iff n = m \vee R\ n\ m$
  - $\neg\texttt{af}\ R_0$ implies $\exists f : \mathbb{N} \to X$ injective and $R\ f_i\ f_j$ for any $i < j$
  - $\neg\texttt{af}\ R_1$ implies $\exists f : \mathbb{N} \to X$ inj. and $\neg R\ f_i\ f_j$ for any $i < j$
  - $\texttt{af}\ R_0$ and $\texttt{af}\ R_1$ implies $\texttt{af}(R_0 \cap R_1)$ hence $\texttt{af}(=_X)$ (absurd)

Higman lemma and the subword relation

- Given $R : \texttt{rel}_2\, X$ over a type X
- The subword relation $<_R^w : \texttt{rel}_2\,(\texttt{list}\, X)$ defined by 3 rules

$$\frac{}{[\,] <_R^w [\,]} \qquad \frac{l <_R^w m}{l <_R^w b :: m} \qquad \frac{a\ R\ b \quad l <_R^w m}{a :: l <_R^w b :: m}$$

- also write $\texttt{subword}\, R$ for $<_R^w$
- Higman lemma (Fridlender 97, non informative version):

$$\texttt{bar}^l\ (\texttt{good}\, R)\ [\,] \quad \text{implies} \quad \texttt{bar}^l\ (\texttt{good}\,(\texttt{subword}\, R))\ [\,]$$

- Nearly the same proof works for $\texttt{bar}_t^l$ instead of $\texttt{bar}^l$
- But this proof cannot be generalized to finite trees...

The product tree embedding, Higman theorem

- Do not confuse with Higman lemma
- trees with same type for all arities: $\texttt{tree}\,X = X \times \texttt{list}(\texttt{tree}\,X)$
- trees of breadth bounded by $k \in \mathbb{N}$:

$$\texttt{btree}\ k\ X = \big\{t \,\big|\, \texttt{tree\_fall}\ (\langle \_|ll\rangle \mapsto \texttt{length}\,ll < k)\ t\big\}$$

- any $t \in T$ is $t = \langle x|t_1, \ldots, t_n\rangle$ with $n < k$, $x \in X$ and $t_i \in T$
- for a relation $R : \texttt{rel}_2\,X$, we define (needs some work...)

$$\frac{s <_R^\times t_i}{s <_R^\times \langle x_n|t_1, \ldots, t_n\rangle} \qquad \frac{x\ R\ y \quad s_1 <_R^\times t_1, \ldots, s_n <_R^\times t_n}{\langle x|s_1, \ldots, s_n\rangle <_R^\times \langle y|t_1, \ldots, t_n\rangle}$$

- Higman theorem: $\texttt{af}_t\,R$ implies $\texttt{af}_t(<_R^\times)$ on $\texttt{btree}\ k\ X$

Higman theorem, an inductive proof

- Type theoretic version of (Veldman 2004)
- $\texttt{tree}(X_n)_{n<k} = T$ where T is lfp of $T = \sum_{n=0}^{k-1} X_n \times T^n$
- one type X_n for each arity $n < k$
- any $t \in T$ is $t = \langle x_n | t_1, \ldots, t_n \rangle$ with $x_n \in X_n$ and $t_i \in T$
- for arity-indexed relations $R : \forall n < k, \texttt{rel}_2\,(X_n)$, we define

$$\frac{s <_R^h t_i}{s <_R^h \langle x_n | t_1, \ldots, t_n \rangle} \qquad \frac{x_n \; R_n \; y_n \quad s_1 <_R^h t_1, \ldots, s_n <_R^h t_n}{\langle x_n | s_1, \ldots, s_n \rangle <_R^h \langle y_n | t_1, \ldots, t_n \rangle}$$

- Higman thm.: $(\forall n < k, \texttt{af}_t\, R_n)$ implies $\texttt{af}_t(<_R^h)$
- by lexicographic induction on $\texttt{af}_t\, R_0 \times \cdots \times \texttt{af}_t\, R_n$

The homeomorphic embedding, Kruskal's theorem

- one type X for all arities: $\texttt{tree}\,X = X \times \texttt{list}(\texttt{tree}\,X)$
- for $R : \texttt{rel}_2\,X$, we define $<^\star_R$ by nested induction

$$\frac{s <^\star_R t_i}{s <^\star_R \langle x_n | t_1, \ldots, t_n \rangle}$$

$$\frac{x_i \; R \; x_j \qquad [s_1, \ldots, s_i] \; (\texttt{subword} <^\star_R) \; [t_1, \ldots, t_j]}{\langle x_i | s_1, \ldots, s_i \rangle <^\star_R \langle x_j | t_1, \ldots, t_j \rangle}$$

- hand-written elimination scheme (nested induction)
- Kruskal theorem: $\texttt{af}_t\,R$ implies $\texttt{af}_t(<^\star_R)$

Kruskal Thm, Tree Embedding upto k

- $\texttt{tree}(X_n)_{n\in\mathbb{N}} = T$ where T is lfp of $T = \sum_{n=0}^{\infty} X_n \times T^n$
- $k \in \mathbb{N}$ and an arity-indexed relation $R : \forall n \in \mathbb{N}, \texttt{rel}_2\,(X_n)$
- one X_n for each arity, but $\boxed{X_k = X_n}$ as soon as $n \geq k$

$$\frac{s <^u_{k,R} t_i}{s <^u_{k,R} \langle x_n | t_1, \ldots, t_n \rangle}$$

$$\frac{n < k \quad x_n\ R_n\ y_n \quad s_1 <^u_{k,R} t_1, \ldots, s_n <^u_{k,R} t_n}{\langle x_n | s_1, \ldots, s_n \rangle <^u_{k,R} \langle y_n | t_1, \ldots, t_n \rangle}$$

$$\frac{k \leq i \quad x_i\ R_k\ x_j \quad [s_1, \ldots, s_i]\ (\texttt{subword}\, <^u_{k,R})\ [t_1, \ldots, t_j]}{\langle x_i | s_1, \ldots, s_i \rangle <^u_{k,R} \langle x_j | t_1, \ldots, t_j \rangle}$$

Kruskal's Tree Theorem, inductive proof

- The recursive statement looks like:

$$\text{if } \mathtt{af}_t\, R_0 \text{ and } \ldots \text{ and } \mathtt{af}_t\, R_k \text{ then } \mathtt{af}_t(<^u_{k,R})$$

- The proof sketch (typed version of Veldman 2004)
  - by induction on lexicographic product $\mathtt{af}_t\, R_0 \times \cdots \times \mathtt{af}_t\, R_k$
  - it is difficult to implement this lexicographic product
  - it is even more difficult with $\mathtt{af}$ instead of $\mathtt{af}_t$
  - Veldman needs Brouver's thesis, but we avoid it
- Kruskal's Tree Theorem: $\mathtt{af}_t\, R$ implies $\mathtt{af}_t(<^\star_R)$
  - use $<^u_{k,R}$ as a lower approximation for $<^\star_R$
  - $<^u_{0,R} \subseteq <^\star_R$ in the case where $n \mapsto R_n$ is constant

Conclusion

- Computational content of $\texttt{af}_t$ or $\texttt{bar}_t^l$ ($\texttt{good}$ R) [ ]
  - a collection of bounds on search-space for good pairs
  - stored in a well-founded tree
- Computational content of theorems:
  - Ramsey thm, Higman's lemma and thm, Kruskal's thm
  - are bound transformation algorithms
- The Coq code: `http://www.loria.fr/~larchey/Kruskal`
  - Free software, available, around 30 000 lines of code
  - Higman's lemma alone below 1000 lines
  - Kruskal's proof complete (both $\texttt{af}$ and $\texttt{af}_t$)
  - but the code can and is still being improved