Partial Computable Functions: Analysis and Complexity

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Goals

- Does the class of partial computable functions have a universal partial computable function?
- What are index set complexity for well-known problems?
- What is a descriptive complexity of images of partial computable functions?

Outline of the Talk

- General framework: Effectively Enumerable Topological Spaces
- Partial Computability over Effectively Enumerable Topological Spaces
- Index set complexity for well-known problems
- Complexity of Images of parial computable functions over computable Polish Spaces

Effectively Enumerable Topological Spaces

Definition. Let $\mathcal{X} = (X, \tau, \alpha)$ be a topological space, where X is a non-empty set, $B \subseteq 2^X$ is a base of the topology τ and $\alpha : \omega \to B$ is a numbering.

Then, $\mathcal{X}$ is effectively enumerable if the following conditions hold.

1. There exists a computable function $g : \omega \times \omega \times \omega \to \omega$ such that

$$\alpha(i) \cap \alpha(j) = \bigcup_{n \in \omega} \alpha(g(i, j, n)).$$

2. The set $\{i \mid \alpha(i) \neq \emptyset\}$ is computably enumerable.

Examples of EE Spaces

- the real numbers with the standard topology;
- the natural numbers with discrete topology;
- computable metric spaces;
- weakly effective ω-continuous domains;
- $C(\mathbb{R})$ with compact-open topology;
- computable Polish spaces;
- .....

Computable Polish Spaces

A computable Polish space $\mathcal{X}$ is

- a complete separable metric space
- without isolated points
- with a countable dense set $\mathcal{B} = \{b_1, b_2, \dots\}$ called a *basis of X*
- with a metric d such that

$$\{(n, m, i) \mid d(b_n, b_m) < q_i,\ \ q_i \in \mathbb{Q}\} \text{ and}$$

$$\{(n, m, i) \mid d(b_n, b_m) > q_i,\ \ q_i \in \mathbb{Q}\} \text{ are computably enumerable.}$$

For a computable Polish space $(X, \mathcal{B}, d)$ in a naturale way we define the numbering of the base of the standard topology as follows. First we fix a computable numbering $\alpha^* : \omega \setminus \{0\} \to (\omega \setminus \{0\}) \times \mathbb{Q}^+$. Then,

$$\alpha(0) = \emptyset,$$

$$\alpha(i) = B(b_n, r) \text{ if } i > 0 \text{ and } \alpha^*(i) = (n, r).$$

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Effectively open sets

Let $\mathcal{X}$ be an effectively enumerable topological space. A set $A \subseteq X$ is effectively open if there exists a computable function $h : \omega \to \omega$ such that

$$A = \bigcup_{n \in \omega} \alpha(h(n)).$$

Partial Computable Functions

Let $\mathcal{X} = (X, \tau_X, \alpha)$ be an effectively enumerable topological space and $\mathcal{Y} = (Y, \tau_Y, \beta)$ be an effectively enumerable T_0–space. A partial function $f : X \to Y$ is called partial computable if the following properties hold. There exist a computable sequence of effectively open sets $\{A_n\}_{n\in\omega}$ and a computable function $h : \omega^2 \to \omega$ such that

1. $dom(f) = \bigcap_{n\in\omega} A_n$ and
2. $f^{-1}(\beta(m)) = \bigcup_{i\in\omega} \alpha(h(m,i)) \cap dom(f)$.

Properties of Partial Computability

Theorem Let $\mathcal{X} = (X, \tau_X, \alpha)$, $\mathcal{Y} = (Y, \tau_Y, \beta)$ and $\mathcal{Z} = (Y, \tau_Z, \gamma)$ be effectively enumerable T_0–spaces.

- Closure under composition:

  If partial functions $f : X \to Y$ and $g : Y \to Z$ are partial computable then $F = g \circ f$ is partial computable.

- Effective Continuity:
  - If $f : \mathcal{X} \to \mathcal{Y}$ is a computable function, then f is continuous at every points of $\mathrm{dom}(f)$.
  - A total function $f : \mathcal{X} \to \mathcal{Y}$ is computable if and only if f is effectively continuous.

Characterisation of Partial Computable Functions over Computable Polish Spaces

Definition (Rogers). A function $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is called enumeration operator if

$$\Gamma_e(A) = B \leftrightarrow B = \{j | \exists i \, c(i,j) \in W_e, \ D_i \subseteq A\},$$

where W_e is the e-th computably enumerable set, and D_i is the i-th finite set.

Theorem. Let $\mathcal{X}$ and $\mathcal{Y}$ be computable Polish spaces.

A function $f : \mathcal{X} \to \mathcal{Y}$ is partial computable if and only if there exists an enumeration operator $\Gamma_e : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that, for every $x \in X$,

1. If $x \in dom(f)$ then $\Gamma_e(\{i \in \omega | x \in \alpha(i)\}) = \{j \in \omega \mid f(x) \in \beta(j)\}$.
2. If $x \notin dom(f)$ then, for all $y \in Y$,

$$\bigcap_{j \in \omega} \{\beta(j) | j \in \Gamma_e(A_x)\} \neq \bigcap_{j \in \omega} \{\beta(j) | j \in B_y\},$$

where $A_x = \{i \in \omega | x \in \alpha(i)\}, \ B_y = \{j \in \omega | y \in \beta(j)\}$.

Majorant-computable Functions

Definition. A partial function $f : \mathcal{X} \to \mathbb{R}$ is called *majorant-computable if the following properties hold. There exist two effectively open sets* $U, V \subseteq X \times \mathbb{R}$ *satisfying requirements:*

1. $\forall x \in X\ U(x)$ *is closed downward and* $V(x)$ *is closed upward;*
2. $f(x) = y \leftrightarrow \{y\} = \mathbb{R} \setminus (U(x) \cup V(x))$*;*
3. $\forall x \in X\ U(x) < V(x)$*.*

To compare the classes of m.-c. functions and real-valued partial computable ones, we need the following notion of weak reduction principle for EE spases.

Weak Reduction Principle

We say that EE space $\mathcal{X}$ meets *weak reduction principle if for any effectively open subsets A, B of X there exists effectively open subsets A_1, B_1 satisfying properties:*

1. $A \setminus B \subseteq A_1 \subseteq A$;
2. $B \setminus A \subseteq B_1 \subseteq B$.

If $\mathcal{X} \times \mathbb{R}$ meets WRP, then $MC_{\mathcal{X}} = \mathcal{PCF}_{\mathcal{X}\,\mathbb{R}}$.
If $MC_{\mathcal{X}} = \mathcal{PCF}_{\mathcal{X}\,\mathbb{R}}$, then $\mathcal{X}$ meets WRP.
So, if the class K of EE spaces is closed under cartesian products, then WRP for K is equivalent to the equality $MC_{\mathcal{X}} = \mathcal{PCF}_{\mathcal{X}\,\mathbb{R}}$ for all spaces $\mathcal{X}$ in K.
We prove WRP for computable metric spaces and find some counterexample in general.

Principal Computable Numbering

For effectively enumerable spaces $\mathcal{X}$ and $\mathcal{Y}$ we denote the set of partial computable function $f : \mathcal{X} \to \mathcal{Y}$ as $\mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$ and nowhere defined function as $\perp$.

A function $\gamma : \omega \times \mathcal{X} \to \mathcal{Y}$ is called computable numbering of $\mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$ if it is a partial computable function and $\{\gamma(n) \mid n \in \omega\} = \mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$ i.e. the sequence of functions $\{\gamma(n)\}_{n\in\omega}$ is uniformly computable. A numbering γ is called principal computable if it is computable and every computable numbering ξ is computably reducible to $\bar{\alpha}$, i.e., there exists a computable function $f : \omega \to \omega$ such that $\xi(i) = \alpha(f(i))$.

Proposition For every computable Polish spaces $\mathcal{X}$ and $\mathcal{Y}$ there exists a principal computable numbering γ of the partial computable functions $f : \mathcal{X} \to \mathcal{Y}$.

Complexity of well-known problems over $\mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$

Theorem. For $\mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$,

- function equality problem: $\{(n, m) | f_n = f_m\}$ is Π^1_1-complete.
- Generalised Rice's Theorem: Let $K \subset \mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$. Then $K \neq \emptyset$ if and only if $Ix(K) \notin \Delta^0_2$.

Unlike previous facts, for many problems related to subclasses of $\mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$ the answer does depend on the choice of Polish spaces $\mathcal{X}$, $\mathcal{Y}$. For example, for $\mathcal{PCF}_{\mathcal{X}\mathbb{R}}$ let us consider totality problem for $\mathcal{X}$, i.e. the set $\{n | \ f_n \text{ is total } \}$.

Proposition.

- Totality problem for reals is Π^0_2-complete.
- Totality problem for Bair space is Π^1_1-complete.

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Borel and analytic subsets of a computable Polish space

- A set B is a Π^0_2–set in the effective Borel hierarchy on $\mathcal{X}$ (a Π^0_2–subset of X) if and only if $B = \bigcap_{n\in\omega} A_n$ for a computable sequence of effectively open sets $\{A_n\}_{n\in\omega}$.
- A set $A \in$ is a Σ^1_1–set in the effective Lusin hierarchy on $\mathcal{X}$ (a Σ^1_1–subset of X) if and only if $A = \{y \mid (\exists x \in X)B(x,y)\}$, where B is a Π^0_2–subset of X^2.

Effectively enumerable T_0–spaces with Point Recovering

Definition. Let $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0–space. We say that $\mathcal{Y}$ admits point recovering if $\{B_x \mid x \in Y\}$ is Σ^1_1-subset of $\mathcal{P}(\omega)$ considered as the Cantor space $\mathcal{C}$, where $B_x = \{n \mid x \in \beta(n)\}$.

Proposition

- Every computable Polish space $\mathcal{X} = (X, \tau, \alpha)$ admits point recovering. Moreover, $\{A_x \mid x \in X\}$ is Π^0_2–subset of $\mathcal{C}$.
- There exists effectively enumerable topological space that does not admit point recovering.

Images of Partial Computable Surjections

Theorem Let $\mathcal{X} = (X, \tau, \alpha)$ be a computable Polish space and $\mathcal{Y} = (Y, \lambda, \beta)$ be an effectively enumerable T_0-space. Then the following assertions are equivalent.

1. There exists a partial computable surjection $f : \mathcal{X} \twoheadrightarrow \mathcal{Y}$.
2. The space $\mathcal{Y}$ admits point recovering.

Complexity of Images of Partial Computable Functions

Theorem Let $\mathcal{X}$ and $\mathcal{Y}$ be computable Polish spaces and $Y_0 \subseteq Y$. Then the following assertions are equivalent.

1. Y_0 is the image of a partial computable function $f : \mathcal{X} \to \mathcal{Y}$.
2. Y_0 is a Σ^1_1–subset of Y.

Outline of the proof.

- Let $\mathcal{X}$ be computable Polish spaces, $\mathcal{Y}$ be an effectively enumerable T_0-space and $Y_0 \subseteq Y$. Then the following assertions are equivalent.
    1. Y_0 is the image of a partial computable function $f : \mathcal{X} \to \mathcal{Y}$.
    2. $\{B_y \mid y \in Y_0\}$ is a Σ^1_1–subset of $\mathcal{C}$.
- Let $\mathcal{Y}$ be a computable Polish space, $Y_0 \subseteq Y$ and $\widetilde{Y}_0 = \{B_y \mid y \in Y_0\}$. Then Y_0 is a Σ^1_1–subset of $\mathcal{Y}$ if and only if $\widetilde{Y}_0$ is a Σ^1_1–subset of $\mathcal{C}$.

Conclusions

Informally, for $\mathcal{PCF}_{\mathcal{X}\mathcal{Y}}$ we showed the following:

- the existence of universal partial computable function and index set complexity for some important problems;
- the existence of a partial computable surjection between any computable Polish space and any effectively enumerable topological space with point recovering;
- descriptive complexity of images of partial computable functions between computable Polish spaces.

Future Work

- Characterisations of complexity of index sets for other important problems on $PCF_{\mathcal{X}\mathcal{Y}}$. We already did few steps in this direction. We showed that for some problems the corresponding complexity does not depend on the choice of a computable Polish space while for other ones the corresponding choice plays a crucial role.
- Characterisations of descriptive complexity of images of total pcf.
- Generalisations of the effective DST on computable Polish spaces to the effective DST on the wider class of effective topological spaces. One of the promising candidates could be effectively enumerable topological spaces with point recovering.

Thank You for your Attention!!!