A Variant of EQU in which Open and Closed Subspaces are Complementary without Excluded Middle

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Background

- Intuitionistic set theory (with powersets)
- PAL $=$ Prime-Algebraic Lattices
  $+$ Scott-continuous functions
- Products $\prod_{i \in I} L_i$;   Exponentials $[L \to M]$
- $\Sigma = (\mathcal{P}\mathbf{1}, \subseteq)$ $0, 1 \in \Sigma$, and possibly more elements
- Scott-continuous $s : \Sigma \to \Sigma$
  is determined by $s\,0$ and $s\,1$

Equilogical Spaces: Definition

- EQU: $X = (L_X, \sim_X)$ where L_X is in PAL
  and '$\sim_X$' is a PER on the points of L_X
- Notation: $|X| = \{a \in L_X \mid a \sim_X a\}$
- $f : X \to Y$ in EQU is
  $f : L_X \to L_Y$ Scott-continuous
  with $a \sim_X b \Rightarrow fa \sim_Y fb$
- $f, g : X \to Y$ are equal in EQU
  if $a \in |X| \Rightarrow fa \sim_Y ga$
- $\mathrm{PAL} \hookrightarrow \mathrm{EQU}$ by $L \mapsto (L, =_L)$

Equilogical Spaces: Properties

- Well-pointed
- Products $\prod_{i \in I} X_i$
  Exponentials $[Y \to Z]$
  Equalizers
- For $p : X \to \Sigma$:
  Open subspace $\mathrm{O}(p) = \{a \in |X| \mid p\,a = 1\}$
  Closed subspace $\mathrm{C}(p) = \{a \in |X| \mid p\,a = 0\}$
- Without EM, neither $\mathrm{O}(p) \cup \mathrm{C}(p) = |X|$
  nor $\mathrm{C}(p) \cup \mathrm{C}(q) = \mathrm{C}(p \wedge q)$ can be shown

Basic Idea for EQU2

- EQU: PER on points, forward morphisms, CCC,
  but open and closed subspaces not complementary
- LOC: Locales defined via opens
  - Morphisms in opposite direction $\longrightarrow$ hard to embed in CCC
    (ELOC uses PERs on generalized points and forward maps)
  - Open and closed sublocales are complementary
    thanks to additional structure ($\wedge$ and $\vee$) on opens
- EQU2: PER on opens of opens (double dual)
  - Forward morphisms $\longrightarrow$ CCC
  - Open and closed subspaces are complementary
    thanks to additional structure

Double Dual

$$\begin{array}{rclcrcl} & L & & & f: & L \to & M \end{array}$$

- In PAL: $\Omega L = [L \to \Sigma]$ $\quad \Omega f : \Omega L \leftarrow \Omega M$

  $\Omega^2 L = [\Omega L \to \Sigma]$ $\quad \Omega^2 f : \Omega^2 L \to \Omega^2 M$

- $\eta_L : L \hookrightarrow \Omega^2 L \qquad \eta_L\, a\, u = u\, a \qquad \Omega^2 f \circ \eta_L = \eta_M \circ f$

- $\vec{\times} : \Omega^2 L \times \Omega^2 M \to \Omega^2 (L \times M)$

  where $A \vec{\times} B = \lambda w^{\Omega(L \times M)}. A\,(\lambda a^L. B\,(\lambda b^M. w\,(a,b)))$

- We also need the “range” $\rho : \Omega^2 L \to [\Sigma \to \Sigma]$

  where $\rho A = \lambda b^{\Sigma}. A(\mathsf{K} b)$

  For all $u : \Omega L$, $\rho A\, 0 \leq A\, u \leq \rho A\, 1$

Restriction of Double Dual

- Problem: $\vec{\times}$, $\Omega^2\pi_1$ and $\Omega^2\pi_2$ are not well related
  Hence CCC cannot be shown
  if PERs on entire $\Omega^2 L$ are used

- Solution:
  Restrict to subset $L^\bullet \subseteq \Omega^2 L$ of “fuzzy points”
  - More than points,
    with additional structure for $\mathrm{O}(p)$ and $\mathrm{C}(p)$
  - Still similar to points $\longrightarrow$ CCC can be shown

Fuzzy Points

- $A : \Omega L \to \Sigma$ is in the image of $\eta_L : L \hookrightarrow \Omega^2 L$
  iff A preserves finite meets and finite (hence all) joins
  iff A preserves empty meet, empty join,
  binary meet, binary join
- $L^\bullet \subseteq \Omega^2 L$:
  Those A that preserve binary meet and binary join
  $A(u \wedge v) = Au \wedge Av \qquad A(u \vee v) = Au \vee Av$
- Points are fuzzy points: $\eta_L : L \hookrightarrow L^\bullet \subseteq \Omega^2 L$
- All constant $K : \Omega L \to \Sigma$ are in $L^\bullet$

Fuzzy Points – Operations

- For $f : L \to M$, $\quad \Omega^2 f : \Omega^2 L \to \Omega^2 M$ restricts to $f^\bullet : L^\bullet \to M^\bullet$
- $\vec{\times} : \Omega^2 L \times \Omega^2 M \to \Omega^2 (L \times M)$ restricts to $(-,-)^\bullet : L^\bullet \times M^\bullet \to (L \times M)^\bullet$
- If $\rho A_1 = \rho A_2$, $\quad$ then $\pi_i^\bullet (A_1, A_2)^\bullet = A_i$
- For $C \in (L \times M)^\bullet$, $\quad (\pi_1^\bullet C, \pi_2^\bullet C)^\bullet = C$
- Not closed under $\wedge$ and $\vee$
- If $s : \Sigma \to \Sigma$ and $A \in L^\bullet \subseteq [L \to \Sigma]$, $\quad$ then $s \circ A \in L^\bullet$

EQU2: Objects

- $(L, \approx)$ where $L \in \mathrm{PAL}$ and $\approx$ PER on $L^\bullet$ such that

  (1) $A \approx B \;\Rightarrow\; \rho A = \rho B$

  (2) For all $s : \Sigma \to \Sigma$: $\quad A \approx B \;\Rightarrow\; s \circ A \approx s \circ B$

  (3) For all constant $K \in L^\bullet$: $\quad K \approx K$

  (4) For all jointly monic $M \subseteq [\Sigma \to \Sigma]$

  (i.e. $(\forall m \in M . \, m\, a = m\, b) \;\Rightarrow\; a = b$):

  $(\forall m \in M . \, m \circ A \approx m \circ B) \;\Rightarrow\; A \approx B$

- Notation: $$|(L, \approx)| \;=\; \{a \in L \mid \eta a \approx \eta a\}$$ $$|(L, \approx)|^\bullet \;=\; \{A \in L^\bullet \mid A \approx A\}$$

EQU2: Morphisms

- Let $X = (L_X, \approx_X)$ and $Y = (L_Y, \approx_Y)$.

  A morphism $f : X \to Y$

  is a continuous function $f : L_X \to L_Y$

  such that $A \approx_X A' \Rightarrow f^\bullet A \approx_Y f^\bullet A'$.

- $f, g : X \to Y$ are equal in EQU2

  if $A \in |X|^\bullet \ \Rightarrow \ f^\bullet A \approx_Y g^\bullet A$

- Global points $x : \mathbf{1} \to X$

  correspond to elements of $|X|$,

  but equality is based on $|X|^\bullet$

  $\longrightarrow$ cannot show that EQU2 is well-pointed

EQU2: Cartesian Closed Category

- $\prod_{i \in I}(L_i, \approx_i) = (\prod_{i \in I} L_i, \approx)$ where $A \approx A'$ iff $\rho A = \rho A'$ and for all i in I, $\pi_i^\bullet A \approx_i \pi_i^\bullet A'$

- For inhabited I, the condition $\rho A = \rho A'$ is redundant

- For empty I: $\mathbf{1} = (\mathbf{1}, \approx)$ where $A \approx A'$ iff $\rho A = \rho A'$ iff $A = A'$

- Exponential $[Y \to Z]$: $L_{[Y \to Z]} = [L_Y \to L_Z]$
  For $H, H' \in L^\bullet_{[Y \to Z]}$, $H \approx_{[Y \to Z]} H'$ iff
  ($\rho H = \rho H'$ and $B \approx_Y B' \Rightarrow @^\bullet(H, B)^\bullet \approx_Z @^\bullet(H', B')^\bullet$)
  where $@ : [L_Y \to L_Z] \times L_Y \to L_Z$

Embedding of PAL into EQU2

- $L \mapsto (L, =_L \bullet)$
- Full subcategory
- Embedding preserves products and exponentials

Subspaces

- Subspace S of $X = (L, \approx)$ is $S \subseteq |X|^\bullet$ such that

  (1) $A \in S \;\; \& \;\; A \approx B \;\; \Rightarrow \;\; B \in S$

  (2) For all $s : \Sigma \to \Sigma$, $\quad A \in S \Rightarrow s \circ A \in S$

  (3) For all constant $K \in L^\bullet$, $\quad K \in S$

  (4) For all jointly monic $M \subseteq [\Sigma \to \Sigma]$,

  $(\forall m \in M . m \circ A \in S) \;\Rightarrow\; A \in S$

- Every subspace S of X induces $X|_S = (L, \approx_S)$ where $A \approx_S B$ iff $A \approx B$ and $A \in S$ (and $B \in S$)

Meets and Joins of Subspaces of X

- Least subspace $\bar{\emptyset}$ is set of constant functions
- Greatest subspace of X is $|X|^\bullet$
- Inhabited meet: $\bigwedge_{i \in I} S_i = \bigcap_{i \in I} S_i$
- Inhabited join: $\bigvee_{i \in I} S_i = \mathcal{M}(\bigcup_{i \in I} S_i)$
  where $\mathcal{M}$ is a closure operator for property (4)
- Subspaces form a frame

Equalizers

- For $f, g : X \to Y$:
  - $\mathrm{E}(f,g) = \{A \in |X|^{\bullet} \mid f^{\bullet}A \approx_Y g^{\bullet}A\}$ is subspace of X
  - $X|_{\mathrm{E}(f,g)}$ is an equalizer of f and g
- Special case $Y = \Sigma$:
  - $\approx_\Sigma$ is equality in $\Sigma^{\bullet}$
  - $f^{\bullet}A =_{\Sigma^{\bullet}} g^{\bullet}A$ iff $Af =_\Sigma Ag$
- Open and closed subspaces: For $p : X \to \Sigma$:
  - $\mathrm{O}(p) = \mathrm{E}(p, \mathsf{K}1) = \{A \in |X|^{\bullet} \mid Ap = A(\mathsf{K}1)\}$
  - $\mathrm{C}(p) = \mathrm{E}(p, \mathsf{K}0) = \{A \in |X|^{\bullet} \mid Ap = A(\mathsf{K}0)\}$

Properties of Open and Closed Subspaces

- $\mathrm{O}(\mathsf{K}\,1) = |X|^{\bullet}$ $\quad\quad$ $\mathrm{C}(\mathsf{K}\,1) = \bar{\emptyset}$
- $\mathrm{O}(p \wedge q) = \mathrm{O}(p) \cap \mathrm{O}(q)$ $\quad\quad$ $\mathrm{C}(p \wedge q) = \mathrm{C}(p) \vee \mathrm{C}(q)$
- $\mathrm{O}(\mathsf{K}\,0) = \bar{\emptyset}$ $\quad\quad$ $\mathrm{C}(\mathsf{K}\,0) = |X|^{\bullet}$
- $\mathrm{O}(\bigvee_{i\in I} p_i) = \bigvee_{i\in I} \mathrm{O}(p_i)$ $\quad\quad$ $\mathrm{C}(\bigvee_{i\in I} p_i) = \bigcap_{i\in I} \mathrm{C}(p_i)$

- $\mathrm{O}(p) \cap \mathrm{C}(p) = \bar{\emptyset}$ $\quad\quad$ $\mathrm{O}(p) \vee \mathrm{C}(p) = |X|^{\bullet}$

Proof of $\mathrm{O}(p) \cap \mathrm{C}(p) = \bar{\emptyset}$

- $\mathrm{O}(p) = \{A \in |X|^{\bullet} \mid A\,p = A\,(\mathsf{K}\,1)\}$
  $\mathrm{C}(p) = \{A \in |X|^{\bullet} \mid A\,p = A\,(\mathsf{K}\,0)\}$
- $\mathrm{O}(p) \cap \mathrm{C}(p) \supseteq \bar{\emptyset}$ is clear.
- For '$\subseteq$', let $A \in \mathrm{O}(p) \cap \mathrm{C}(p)$.
- Then $A\,p = A\,(\mathsf{K}\,0)$ and $A\,p = A\,(\mathsf{K}\,1)$.
- Hence $A\,(\mathsf{K}\,0) = A\,(\mathsf{K}\,1)$,
  so A is constant and thus in $\bar{\emptyset}$.

Proof of $\mathrm{O}(p) \vee \mathrm{C}(p) = |X|^\bullet$

- $\mathrm{O}(p) \vee \mathrm{C}(p) \subseteq |X|^\bullet$ is clear. For '$\supseteq$', let $A \in |X|^\bullet$.
- Let $s_0, s_1 : \Sigma \to \Sigma$, $s_0\,a = a \vee A\,p$, $s_1\,a = a \wedge A\,p$
- Recall $\mathrm{C}(p) = \{B \in |X|^\bullet \mid B\,p = B\,(\mathsf{K}\,0)\}$.
  $$(s_0 \circ A)\,(\mathsf{K}\,0) = s_0\,(A\,(\mathsf{K}\,0)) = A\,(\mathsf{K}\,0) \vee A\,p = A\,p$$
  $$(s_0 \circ A)\,p = s_0\,(A\,p) = A\,p \vee A\,p = A\,p$$
- Hence $s_0 \circ A \in \mathrm{C}(p) \subseteq \mathrm{O}(p) \vee \mathrm{C}(p)$.
- In a similar way, $s_1 \circ A \in \mathrm{O}(p) \subseteq \mathrm{O}(p) \vee \mathrm{C}(p)$.
- $\{s_0, s_1\}$ is jointly monic since in every distributive lattice
  $$a \vee c = b \vee c \;\;\&\;\; a \wedge c = b \wedge c \;\Rightarrow\; a = b.$$
- Property (4) gives $A \in \mathrm{O}(p) \vee \mathrm{C}(p)$.

Conclusion

- • Definition of EQU2, a variant of EQU
- + EQU2 is a CCC (like EQU)
- + In EQU2, open and closed subspaces are complementary even without Excluded Middle (not true for EQU)
- – EQU2 is more complicated than EQU
- – EQU2 is not necessarily well-pointed (but EQU is)
- ! With Excluded Middle, EQU2 and EQU are isomorphic categories