Geometric Lorenz attractors are computable

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Introduction

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- **But also many questions**: what happens for dimensions ≥ 3?

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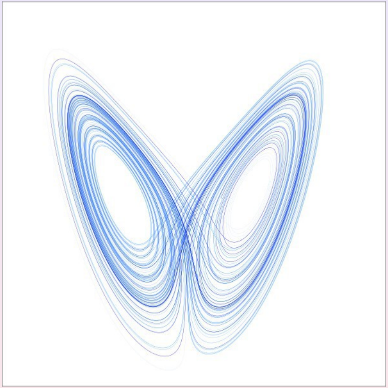
- This approach led to a new understanding of the richness of behaviours of dynamical systems.
- Most notably these computer simulations provided evidence that new types of robust attractors other than equilibria and periodic orbits could exist: the strange attractors.
- The most iconic of such attractors is the Lorenz attractor, first described by E. Lorenz in 1962.
- There is a recent series of works which show that Lorenz-like attractors are fairly typical for large classes of systems defined in $\mathbb{R}^3$.

The Lorenz attractor



$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \end{cases}$$

Classical values for parameters: $\sigma = 10, \rho = 28, \beta = 8/3$

But is the Lorenz attractor real?

Problem (S. Smale)

Does the Lorenz attractor exist?

- Perhaps the images of Lorenz attractors are just the result of the cumulation of roundoff errors?
- Can we rigoursly prove it exist?
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- This problem was solved in 2002 by W. Tucker using a combination of rigourous numerics and normal form theory.

What about computability?

- In various applications it is useful to know something about the asymptotic behaviour of a system (e.g. in verification, etc.) in an automated manner
- It is not always the case that we can compute this behaviour because often this reduces to solving the Halting problem
- But what about the case of smooth three-dimensional flows?

Question

Is the Lorenz attractor computable?

Some preliminaries

- In the late 1970s several authors suggested the use of geometrical Lorenz models to better understand the Lorenz attractor.
- Such models were assumed to have the qualitative behaviour which was numerically observed on the Lorenz system.
- It was soon shown that geometrical Lorenz models have a strange attractor with properties compatible to those observed via numerical experiments.
- W. Tucker essentially showed (using rigourous numerics and normal form theory) that the Lorenz system behaves like a geometric Lorenz model, thus supporting a strange attractor.

Our result

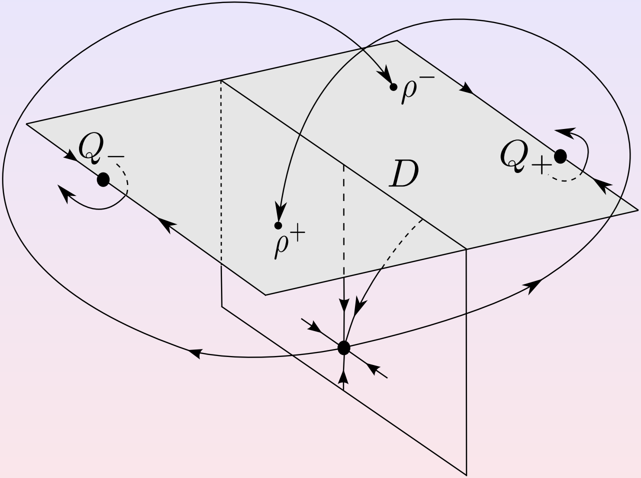
Theorem

Let ϕ be the a (C^2) flow of some Lorenz geometric system. Then:

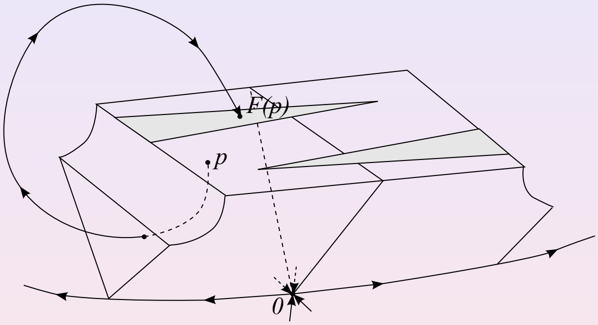
1. *The global attractor $\mathcal{A}$ of a geometric Lorenz flow ϕ is computable from a (C^2) name of ϕ.*
2. *The geometric Lorenz flow admits a physical measure which is computable from a (C^2) name of ϕ.*

The geometric Lorenz model

- A geometric Lorenz model has three equilibrium points: the origin, Q_-, and Q_+.
- The origin is a saddle point: its stable manifold is the yz-plane while its unstable manifold intersects the plane $z = 27$ from above at two points $\rho^+ = (r^-, t^-)$ and $\rho^- = (r^+, t^+)$.
- Both Q_- and Q_+ lie on the plane $z = \rho - 1 = 27$. Their stable lines are parallel to the y-axis, and the flow near these points rotates around their stable lines.
- Let Σ be a rectangle contained in the plane $z = 27$ such that $\rho^\pm$ is contained in Σ, the two opposite sides of Σ parallel to the y-axis pass through the equilibrium points Q_- and Q_+, and these two sides form portions of the stable lines at Q_- and Q_+.
- Let D be the intersection of the yz-plane and Σ.

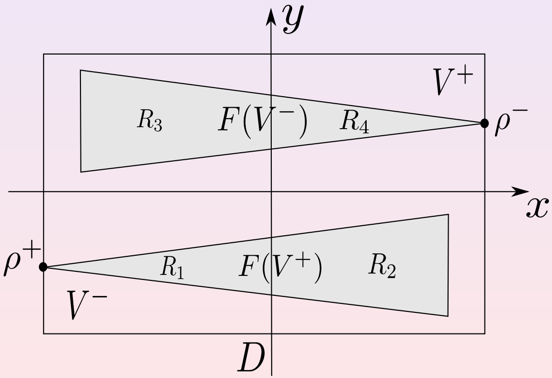

ρ^-
Q_-
Q_+
D
ρ^+

- Σ is a cross section for the flow;
- All trajectories go downwards through Σ;
- All trajectories originating in Σ and not entering D spiral around Q_- or Q_+ and return to Σ as time moves forward;
- All trajectories beginning at points in D tend to the origin as time moves forward and never return to Σ;
- This implies that there is a Poincaré return map $F : \Sigma_- \bigcup \Sigma_+ \to \Sigma$, where $\Sigma_- = \{(x, y) \in \Sigma | x < 0\}$ and $\Sigma_+ = \{(x, y) \in \Sigma | x > 0\}$.


F(p)
p
0

(3D picture of the Lorenz attractor)

- Let $V = \{(x,y) | r^- \leq x \leq r^+, -27 \leq y \leq 27\}$ (the number 27 is arbitrarily chosen; other positive numbers can be used as well);
- The Lorenz flow also has the property that all points in the interior of $\Sigma \setminus D$ have a trajectory which will eventually reach V and $F(V \setminus D) \subseteq V$. Thus we can restrict the analysis of the flow to V.



Main characteristics

(F-1) The set $\mathcal{F}$, $\mathcal{F} = \{x{=}\text{constant}\}$, is invariant under the action of F. In other words, the x-coordinate of the image $F(x_0, y_0)$ depends only on x_0.

(F-2) There are functions f and g such that F can be written as

$$F(x, y) = (f(x), g(x, y)) \quad \text{for } x \neq 0$$

and $F(-x, -y) = -F(x, y)$.

(F-3) $f'(x) > \sqrt{2}$ for $x \neq 0$ and $f'(x) \to \infty$ as $x \to 0$; $0 < f(r^+) < r^+$ and $r^- < f(r^-) < 0$ (recall that the unstable manifold of the origin first intersects V from above at points ρ^+ and ρ^-).

(F-4) $0 < \partial g/\partial y \leq c < 1/\sqrt{2}$ and $0 < \partial g/\partial x \leq c$ for $x \neq 0$ and $\partial g/\partial y \to 0$ as $x \to 0$. Without loss of generality, c can be assumed to be a rational number and $\partial g/\partial y \to 0$ to be monotonic as $x \to 0$.

A consequence of (F-2)-(F-4) is that:

(F-5) $\lim_{x \to 0^-} F(x, y) = (r^+, t^+)$ and $\lim_{x \to 0^+} F(x, y) = (r^-, t^-)$, where $\rho^- = (r^+, t^+)$ and $\rho^+ = (r^-, t^-)$. The symmetry property (F-2) implies that $r^- < 0 < r^+$ and $r^- = -r^+$.

Main results

Theorem

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Proof of the result

Let us show the computability of $\mathcal{A}$.

- The first step is to consider a reduced problem where we show uniform computability of $A = \mathcal{A} \cap V$.

Proposition

The operation $(F, \rho^{\pm}) \to A$ *is computable.*

It can be shown that

$$A = \bigcap_{n \geq 0} \overline{F^n(V \setminus D)}$$

so we take $A_n = \overline{F^n(V \setminus D)}$ and show that:

i) the sequence $\{A_n\}$ is computable from F and $\rho^{\pm}$;

ii) $\max_{(x,y)\in V} |d_{A_{n+1}}(x,y) - d_{A_n}(x,y)| \leq 108c^n$ (see (F-4) for the definition of the number c);

iii) thus the computable sequence $\{d_{A_n}\}_{n\in\mathbb{N}}$ converges to a computable function d_A;

iv) since d_A is computable, then so is A

This lemma is not obvious due to the presence of the “singularity” line D where the return map is not defined.

Lemma

Let ϕ be the flow of some Lorenz geometric system. Then we can uniformly compute from a (C^2) name of ϕ:

1. *The return function F (and its components f, g).*
2. *The return time function $r : V \backslash D \to [0, +\infty)$.*
3. *The points $r^{\pm}, t^{\pm}$.*

This lemma + the previous proposition show that A is computable from the flow ϕ. From the planar projection A, it is not too dificult to show the computability of the (whole) attractor $\mathcal{A}$.

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- Not yet!
- The problem has to do with the fact that we need a constructive version of Tucker's proof.
- In particular it is not enough to show that a foliation exists for the Lorenz attractor.
- We still have to show that a computable foliation of the Lorenz attractor exists which can be computably mapped into the standard foliation $\mathcal{F} = \{x = \text{constant}\}$ with the properties F1–F5.

Thank you!