Ramsey actions and Gelfand duality in logic

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1 Historical Background and Motivation

The work can be viewed, with the benefit of hindsight, especially because of the work by Kechris, Pestov and Todorcevic (2005), as an interpretation of the ideas in the papers by Blass (1987) and Coquand (1992).

In both these earlier papers, Ramsey theory plays a prominent role and they deal with dynamical versions of Ramsey theorems and its implications for mathematical logic.

In the case of Blass, the focus is on the tension between the axiom of choice and the Boolean Prime Ideal theorem (each locale has a point) and where he accentuated the rôle of the symmetry aspects of traditional Ramsey theory by introducing the concepts of

`Ramsey groups and Ramsey actions.`

Coquand addresses the problem of understanding the topological versions of Ramsey theorems from the viewpoint of constructive mathematics and proposes the problem of viewing these phenomena from a suitable point-free theory of topological dynamical systems.

Along these lines one might be able to attain a beter understanding of the algorithmic and algorithmically random content of the topological versions of Ramsey theorems. An initial exploration of this was undertaken by F (2011-2012).

During 1995-1999 F wrote a sequence of papers identifying the rôle that symmetry plays in identifying Ramsey objects in a class of finite combinatorial configurations. These results rely heavily on the work by Nesetril and Rödl in combinatorics and by Abramsky and Harrington in model theory.

Some aspects of these results were placed within a much broader perspective by the paper of Kechris, Pestov and Todorcevic (2005).

In very broad terms they accomplished the following:

Let X be the Fraïssé limit of a Fraïssé age of a first order structure

and write G for its symmetry group and topologise it by viewing it as having the topology induced by being a subgroup of the symmetry group S_∞ of the countable set X,

the group S_∞ thus having the pointwise convergence topology.

The objects in the age of X are all "Ramsey objects" (a purely combinatorial notion) iff all topological actions of G on some compact Hausdorff space have fixed points. (In the language of dynamical systems, this expresses the fact that G is an extremely amenable group.)

I looked at these results from the viewpoint of C^*-algebras in order to find a representation of topological Ramsey results in a categorically invariant manner.

With the benefit of hindsight, one can say that a closed subgroup of S_∞ is extremely amenable iff it has the property, in the language of Blass (1987), that all transitive actions of G on discrete spaces are "Ramsey actions".

In this project, we discuss how we can understand these results in a categorical duality framework and indeed within Gelfand duality and Stone duality. The idea of looking at these results from the angle of Gelfand duality was suggested to me upon reading a paper by Glasner and Weiss (2003) on the symmetries of the Cantor ternary set.

We show that these investigations lead to problems of independent interest especially towards understanding hyperstonean spaces, the Gelfand duals of commutative von Neumann algebras, from the viewpoint of Stone duality.

This is unexplored territory, but I have indicated that ideas from quantum logic, to wit , Gleason's theorem, suitably generalised, can shed some light on this problem.

We propose that the paper by Freyd (1992)

(All topoi are localic, or Why permutation models prevail)

must also shed light on these problems especially as far as the challenge, proposed by Coquand, is concerned.

2 A concrete example of ideas involved

If G is a Hausdorff topological group, we write $C_{rub}(G)$ for the commutative C^*-algebra with an identity element based on the bounded (complex-valued) functions on G which are right-uniformly continuous.

Thus a function $f : G \to \mathbb{C}$ belongs to $C_{rub}(G)$, iff it is bounded and for every $\epsilon > 0$, there is some symmetric neighbourhood V of the identity element of G (meaning that $V = V^{-1}$), such that:

$$s^{-1}t \in V \Longrightarrow |f(s) - f(t)| < \epsilon.$$

The Gelfand dual of $C_{rub}(G)$ is denoted by Γ_G, which is a compact Hausdorff space.

(Thus, for example, if G is a discrete space then $C_{rub}(G)$, as a C^*-algebra, can be identified with $\ell^\infty(G)$ and Γ_G with $\beta(G)$, the Stone-Cech compactification of G. In fact, $\ell^\infty(G)$ is a von Neumann-algebra which means that Γ_G is in fact a "hyperstonean space".)

We shall refer to fixed points of the action of G on Γ_G as *Ramsey characters*.

To motivate this terminology, let as look at the following different way of looking at the oldest result in Ramsey theory. In a way, we present a dynamical C^*-algebraic reformulation of this result.

But first we must introduce some terminology.

Let η be the Cantor order. This means it is an example of a countable model of the first order properties of the structure $(\mathbb{Q}, \leq)$.

Write S_∞ for the symmetry group of a countably infinite set. Without loss of generality, we may assume that the countable infinite set on which S_∞ acts is coded by the natural numbers $\mathbb{N}$. As such we can view S_∞ as a subset of $\mathbb{N}^\mathbb{N}$. We topologise $\mathbb{N}^\mathbb{N}$ by imposing the discrete topology and then a product topology.
The resulting space is frequently referred to as the Baire space . We topologise S_∞ via encodings to view S_∞ as embedded thus

$$S_\infty \subset \mathbb{N}^\mathbb{N}.$$

As such it is a closed subgroup of the Baire space $\mathbb{N}^\mathbb{N}$.

We have, writing $G = \texttt{Aut}\ \eta$, that Γ_G is a Stonean space. Indeed, C^*-algebraically:

$$C_{rub}(G) \simeq C(\varprojlim_{H <_o G} \beta(G/H)) \simeq \varinjlim_{H <_o G} \ell^\infty(G/H), \tag{1}$$

and hence, by Gelfand duality, we have the topological homeomorphism

$$\Gamma_G \simeq \varprojlim_{H <_o G} \beta(G/H). \tag{2}$$

Here $H <_o G$ means that H is an open subgroup of G and for a discrete space D , we write $\beta(D)$ for the Stone-Čech compactification of D.

The topological homeomorphisms are not G-dynamical isomorphisms.

Note that the second isomorphism in (1) expresses $C_{rub}(G)$ as a direct limit of Von Neumann algebras.

Let us recall the

Oldest Ramsey Theorem. (Ramsey 1932) *For natural numbers r, n, k there is a natural number N , such that for any r-colouring χ of the k-subsets of $[N] := \{1, \ldots, N\}$, there is a a n-subset A of $[N]$ such that χ assumes a constant value on all the k-subsets of A.*

A case can be made for the statement that this classical finitary Ramsey theorem can be expressed, in the context of C^*-algebras as

Theorem 1 *Let* $\mathtt{Aut}(\eta)$ *be the topological symmetry group of the Cantor order* η*. Write* C *for the* C^**–algebra of right-uniformly continuous functions on* $\mathtt{Aut}\ \eta$*. Then there is a Gelfand character* χ *on* C *such that*

$$\sigma\chi = \chi,$$

for all $\sigma \in \mathtt{Aut}\ \eta$*.*

In particular, $G = \mathtt{Aut}\ \eta$ admits a "Ramsey character" χ.

In this project we explore, among other things, the extent to which such a Ramsey character is "random" or could be constructively expressed.

This work is a continuation of what can be found in F (1996-1999) and (2011-).

This investigation leads us to exploring the Stonean structure of Gelfand duals of von Neumann algebras.

We shall also relate this statement to permutation models in set theory, both within classical set theory and Grothendieck toposes.

The envisaged goal of this project is to understand dynamical versions of Ramsey theorems and its implications for logic in a constructive and/or effective topological and probabilistic context.

Following Blass , we introduce the notion of a *Ramsey action*. Let G be a topological group and X a discrete space. A continuous action $G \times X \to X$, denoted by $(g, x) \to gx$ of G on X, is said to be a Ramsey action iff the following holds:

Let $\chi : X \to r$ be any r-colouring of X. Let F be any finite set of X. Then there is some $\sigma \in G$, such that χ is monochromatic on the translate σF.

Note that a Ramsey action is necessarily transitive. If not, distribute F over two disjoint orbits of the action of G on X and give the two orbits in X different colours and colour the other orbits arbitrarily.

Let $\mathcal{L}$ be the signature of a first-order structure and let $\mathbf{K}$ be the age of some countable $\mathcal{L}$-structure. For $A, \pi \in \mathbf{K}$ we denote by A^π the set of all the (model-theoretic) structure-preserving embeddings of π in A.

For a natural number $r \geq 1$ and for $\pi, A, B \in \mathbf{K}$ we introduce the predicate $B \rightsquigarrow (A)^\pi_r$ (a variant of the Erdős-notation) to mean:

$$B \rightsquigarrow (A)^\pi_r \iff \Big(\forall\ B^\pi \xrightarrow{\chi} r\ \exists\ A \rightarrowtail^{\alpha} B \quad \begin{array}{ccc} A^\pi & \xrightarrow{\alpha_*} & B^\pi \\ & \searrow^{!} \quad \oplus \quad \swarrow_{\chi} & \\ & r & \end{array} \Big).$$

Here $\alpha_* : A^\pi \to B^\pi$ is the mapping that takes an embedding

$$\pi \overset{x}{\rightarrowtail} A$$

to the induced embedding

$$\pi \overset{\alpha x}{\rightarrowtail} B\ .$$

The predicate says that some fibre of each $\chi : B^\pi \to r$ will contain a copy of A^π, to wit $\alpha_*(A^\pi)$ as given by the embedding α of A into B.

In other words, in a more elementary language, $B \rightsquigarrow (A)^\pi_r$ iff: for every r-colouring χ of the set B^π consisting of the embeddings of π in B (copies of π in B), there is an embedding α of A into B such that $\chi\alpha_*$ is a constant. This means that χ assumes a constant value on all the embeddings of π into the image $A' = \alpha_*(A) \subset B$ of A under α.

We shall call an age $\mathbf{K}$ a *Ramsey age* if, for all $\pi, A \in \mathbf{K}$ with $A^\pi \neq \emptyset$, and all natural numbers $r \geq 1$, there is some $B \in \mathbf{K}$ such that $B \rightsquigarrow (A)^\pi_r$.

A somewhat broader outlook:

For those who doubt the value of studying ZFA, it might be interesting to look at the paper by Freyd (1992). I suspect these results of Freyd bring us closer to the challenge proposed by Coquand.

Indeed, paraphrasing Freyd: A full subcategory $\mathcal{V}$ in a Grothendieck topos $\mathcal{A}$ is an *exponential variety* if it is closed under the formation of subobjects, cartesian products and power-objects.

The category $\mathcal{V}$ is necessarily a topos, its inclusion functor is logical and has both adjoints.

Given any object A in $\mathcal{A}$, the minimal exponential variety containing A may be constructed as the full subcategory of all subobjects that appear in the transfinite sequence recursively defined by

$$P_0 = A,\ P_\alpha = P(\sum_{\beta<\alpha} P_\beta)$$

.

All exponential varieties so arise. $\mathbf{V}_\emptyset$ is called the well-founded part of $\mathbf{A}$.

3 Why permutation models prevail: d'aprés Freyd

Theorem 2 *(Freyd). There is some* `single` *Boolean topos $\mathcal{B}$ such that for every Grothendieck topos $\mathcal{A}$, there is a locale $\mathcal{L}$ in $\mathcal{B}$ such that $\mathcal{A}$ appears as an exponential variety in the topos of $\mathcal{L}$-sheaves over $\mathcal{B}$.*

For a topological group G, let $B(G)$ be the category of continuous actions of G on discrete spaces. For either $G = S_\infty$ or $G = \mathtt{Aut}(\eta)$, where η is the Cantor order, the Boolean topos $\mathcal{B} = B(G)$ will meet with the requirements of Freyd's theorem.

It is an interesting problem to characterise the topological groups G for which this will be the case.

As Freyd pointed out, such groups will be, in more modern language, non- Archimedean and Polish, thus necessarily isomomorphic to closed subgroups of S_∞, being thus essentially automorphism groups of Fraïssé limits.

He didn't say this in these words, but work by Kechris and others, together with some model theory, enable one to make such a statement.

4 Summary

The discussion is based on the following observation:

Theorem 3 *Let G be a Polish nonarchimedean group, i.e, isomorphic to a closed subgroup of S_∞. The following statements are equivalent:*

1. *With G we can build a permutation (Mostowski-Fraenkel-Specker) model of ZFA (Zermelo-Fraenkel with Atoms) in which AC (Axiom of Choice) is false but the ultrafilter theorem (or, in ZF, every locale has a point) is true.*

2. *Every transitive and continuous action of G on a discrete set X is a Ramsey action.*

3. *G is the automorphism group of the Fraïssé limit of a Ramsey age whose members are rigid (having no non-trivial automorphisms).*

4. *The C^*-algebra $C_{rub}(G)$ of bounded right-uniformly continuous (complex-valued) functions on G admits a Gelfand character χ which is G-invariant.*
   (We call such a character χ a Ramsey character).

5 Random structure of Ramsey characters.

Definition 1 *(F 2015) A topological group G is* fecund *if it contains an* open *subgroup H such that $(G:H)$ is infinite. Such an H will be called a witness to the fecundity of G.*

Theorem 4 *Let $\mathcal{L}$ be a countable first order language and T a first-order theory in $\mathcal{L}$ which is complete in the sense that for each sentence ϕ of $\mathcal{L}$* `exactly` *one of ϕ or $\neg\phi$ is a first-order logical consequence of T. Suppose any two countable models of the theory T are isomorphic, that is, we suppose that the theory T is $\aleph_0$-categorical. Then if A is any countable model of the theory T, the group $G = \mathtt{Aut}(\mathtt{A})$, when topologised as an embedded subgroup of S_∞, is fecund. In fact, for every $n \in \mathbb{N}$, there will be an n-tuple $\mathbf{a}$ over A such that the orbit $G\mathbf{a}$ is (countably) infinite. If H is the stabiliser of such an n-tuple $\mathbf{a}$, then H is an open subgroup of G, which witnesses the fecundity of G.*

For $g \in G$, let $\delta_g : C_{rub}(G) \longrightarrow \mathbb{C}$ be the Gelfand character given by $\delta_g(f) = f(g)$ for all $f \in C_{ub}(G)$. The mapping

$$\Delta : G \longrightarrow \Gamma_G$$

is given by $g \mapsto \delta_g$. The mapping is clearly continuous. Moreover

$$\sigma\Delta(g) = \Delta(\sigma g),$$

for all $\sigma, g \in G$.

Theorem 5 *(F, de Beer 2015 -2017)*
Let G be a Polish non-archimedean group. Then $\Delta : G \longrightarrow \Gamma_G$ is injective and $\Delta(G)$ is dense in Γ_G. Moreover, this mapping induces a Borel isomorphism between G and $\Delta(G)$.

Theorem 6 *(F, de Beer 2016) Let G be a fecund Polish non-Archimedean topological group. Let ν be a probability measure on Γ_G which is G-invariant. Then*

$$\nu(\Delta(G)) = 0.$$

No Ramsey character (a fixed point of the action of G on Γ_G) associated with G will be an element of $\Delta(G)$.

This result suggests that each Ramsey character χ is in some sense “random”. It is outside a specific set $\Delta(G)$ which is negligible with respect to any G-invariant Radon probability measure on Γ_G even though this set is dense in Γ_G. Moreover $\Delta(G)$ is Borel equivalent to G, topologically viewed and is in fact a Borel set. (Not trivial.)

However, the following result places a restriction on this suggestion:

Theorem 7 *F, de Beer, 2016-2017) Let G be a Hausdorff topological group such that Γ_G is hyperstonean. This means essentially that $C_{rub}(G)$ is a commutative Von Neumann algebra. (This is, for example, the case when G is discrete.) Then there is a Radon probability measure μ on Γ_G such that $\mu(\Delta(G)) > 0$. If G is moreover Polish, non-Archimedean and fecund, such a measure will necessarily not be G-invariant.*

Towards understanding hyperstonean spaces, the following remarks might be useful: If $\mathbb{L}$ is a lattice having a smallest element 0 and largest element 1 we write $\mathcal{Q}(\mathbb{L})$ for the family of ultrafilters (maximal filters) on $\mathbb{L}$. We topologise $\mathcal{Q}(\mathbb{L})$ by the topology generated by the sets of the form $\mathcal{Q}(\mathbb{L})_a$, for some $a \in \mathcal{Q}(\mathbb{L})$, where $\mathcal{Q}(\mathbb{L})_a$ consists all those ultrafilters that has a as an element. When $\mathbb{L}$ is a Boolean algebra, then $\mathcal{Q}(\mathbb{L})$, thus topologised, is exactly the Stone topology associated with the lattice. For this reason, we shall refer to $\mathcal{Q}(\mathbb{L})$ as the Stone space on $\mathbb{L}$.

Theorem 8 *(de Groote 2005, 2011). Let $\mathcal{R}$ be an abelian von Neumann algebra. Let $\mathbb{L}$ be the projection lattice of $\mathcal{R}$. Then the Gelfand dual of $\mathcal{R}$ is homeomorphic to the Stone space $\mathcal{Q}(\mathbb{L})$.*

Every hyperstonean space Γ can be viewed as a Stone space directly topologically definable within Γ as follows:

$$\Gamma \rightsquigarrow \mathcal{R} := C(\Gamma) \rightsquigarrow \mathbb{L} = P(\mathcal{R}) \rightsquigarrow \mathcal{Q}(\mathbb{L}) \simeq \Gamma,$$

or

$$\texttt{hyperstonean space } \Gamma \rightsquigarrow \texttt{abelian von Neumann algebra } \mathcal{R}$$

$$\rightsquigarrow \texttt{projection lattice } \mathbb{L} \textit{ of } \mathcal{R} \rightsquigarrow \texttt{Stone space } \mathcal{Q}(\mathbb{L}) \simeq \Gamma,$$

Proof: The fact that Γ is hyperstonean means exactly that $\mathcal{R} = C(\Gamma)$ is a von Neumann algebra. Of course, Γ is the Gelfand dual of the C^*-algebra $\mathcal{R}$. It follows from the result of de Groote that this dual must be homeormorphic to $\mathcal{Q}(\mathbb{L})$ with the Stone topology, where $\mathbb{L}$ is the projection lattice of $\mathcal{R}$.

A dual picture might be:

$$\mathcal{R} \rightsquigarrow \mathbb{L} := P(\mathcal{R}) \rightsquigarrow \Gamma := \mathcal{Q}(\mathbb{L}) \rightsquigarrow C(\Gamma) \simeq \mathcal{R}.$$

or

$$\texttt{commutative von Neumann } \mathcal{R} \rightsquigarrow \texttt{projection lattice } \mathbb{L}$$
$$\rightsquigarrow \texttt{Stone space } \Gamma = \mathcal{Q}(\mathbb{L}) \rightsquigarrow \mathcal{R} \simeq C(\Gamma).$$

The crucial observation here is that the Gelfand dual Γ of an abelian von Neumann algebra $\mathcal{R}$ is indeed the Stone space associated with the the projection lattice associated with $\mathcal{R}$. This result is fairly recent and was first established by Hans de Groote (2005). He gave an alternative and shorter proof in 2012 where it is deduced from the powerful generalised Gleason theorem in quantum logic .