σ-locales and Booleanization in Formal Topology

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EU, planet Earth, Solar system, Milky Way . . .

σ-frames and σ-locales

(see Alex Simpson's talk)

A σ-**frame** is a poset with:

- countable joins (including the empty join)
- and finite meets (including the empty meet)

in which binary meets distribute over countable joins.

σ**Loc** = category of σ-frames and the opposite of σ-frame homomorphisms

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Aim of this talk:

to prove some facts about σ-frames

in a constructive and predicative framework, namely Formal Topology,

(which can be formalized in the Minimalist Foundation + AC_ω).

But, what is a countable set? (constructively)

Some classically equivalent definitions for a set S:

- S is either (empty or) finite or countably infinite;
- S is either empty or enumerable;
- Either $S = \emptyset$ or there exists $\mathbb{N} \twoheadrightarrow S$ (onto).
- ...

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Definition

S is *countable* if there exists $\mathbb{N} \rightarrow 1 + S$ with S contained in the image
(see literature on Synthetic Topology: Andrej Bauer, Davorin Lešnik).

S is countable $\Longleftrightarrow$ there exists $D \twoheadrightarrow S$ with $D \subseteq \mathbb{N}$ detachable
(see Bridges-Richman *Varieties...* 1987).

The set of countable subsets

Given a set S, we write $\mathcal{P}_{\omega_1}(S)$ for the set of countable subsets of S.

$$\mathcal{P}_{\omega_1}(S) \quad \cong \quad (1+S)^{\mathbb{N}}/\sim$$

where $f \sim g$ means $S \cap f[\mathbb{N}] \; = \; S \cap g[\mathbb{N}]$.

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Some properties of $\mathcal{P}_{\omega_1}(S)$

- $\mathcal{P}_{\omega_1}(S)$ is closed under countable joins (AC_ω).
- If equality in S is decidable, then $\mathcal{P}_{\omega_1}(S)$ is a σ-frame.
- $\mathcal{P}_{\omega_1}(1)$ = "open" truth values (Rosolini's dominance)
  = free σ-frame on no generators
  = terminal σ-locale.

σ-locales *in* Formal Topology

Let L be a σ-locale.

For $a \in L$ and $U \subseteq L$ define

$$a \lhd_L U \quad \overset{def}{\Longleftrightarrow} \quad a \leq \bigvee W \text{ for some countable } W \subseteq U.$$

$\lhd_L$ is a **cover relation** (Formal Topology), that is,

$$\frac{a \in U}{a \lhd U} \qquad \frac{a \lhd U \quad \forall b \in U . b \lhd V}{a \lhd V} \qquad \frac{a \lhd U}{a \wedge c \lhd \{b \wedge c \mid b \in U\}} \qquad \frac{}{a \lhd \{\top\}}$$

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Proposition

$(L, \lhd_L, \wedge, \top)$ is (a predicative presentation of) the free frame over the σ-frame L.

(cf. Banashewski, *The frame envelope of a σ-frame*, and Madden, *k-frames*)

Lindelöf elements in a frame

An element a of a frame F is **Lindelöf** if for every $U \subseteq F$

$$a \leq \bigvee U \quad \Longrightarrow \quad a \leq \bigvee W \text{ for some countable } W \subseteq U.$$

Lindelöf elements are closed under countable joins (not under finite meets, in general).

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σ-coherent frame $=$

- Lindelöf elements are closed under finite meets (and hence they form a σ-frame), and
- every element is a (non necessarily countable) join of Lindelöf elements.

σ-coherent formal topologies

σ-coherent frames can be presented as formal topologies $(S, \lhd, \wedge, \top)$ where

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Proposition

Given a σ-locale L,

$(L, \lhd_L, \wedge, \top)$ is σ-coherent and

its σ-frame of Lindelöf elements is L

So σ-locales can be seen as σ-coherent formal topologies

(with a suitable notion of morphism).

Examples

Examples of σ-coherent formal topologies:

point-free versions of

- Cantor space $2^{\mathbb{N}}$
- Baire space $\mathbb{N}^{\mathbb{N}}$
- $S^{\mathbb{N}}$ with S countable.

So their Lindelöf elements provide examples of σ-locales.

Dense sublocales

A **congruence** $\sim$ on a frame L is
an equivalence relation compatible with finite meets and arbitrary joins.

The quotient frame $\boxed{L/\sim}$ *is* a sublocale of L.

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$L/\sim$ is **dense** if $(\forall x \in L)(x \sim 0 \;\Rightarrow\; x = 0)$

Some well-known fact about dense sublocales:

- the "intersection" of dense sublocales is always dense (!), hence
- every locale contains a smallest dense sublocale
- which turns out to be a complete Boolean algebra ("Booleanization");
- the corresponding congruence $x \sim y$ is $\boxed{\forall z(y \wedge z = 0 \Longleftrightarrow x \wedge z = 0)}$

Boolean locales are good but...

- non-trivial discrete locales are never Boolean
- Boolean locales have no points
- non-trivial Boolean locales are never *overt*

unless your logic is classical!

Recall that $(S, \lhd)$ is **overt** if there exists a predicate *Pos* such that

$$\frac{Pos(a) \quad a \lhd U}{\exists b \in U. Pos(b)} \qquad \frac{a \lhd U}{a \lhd \{b \in U \mid Pos(b)\}}$$

INTUITION: $Pos(a)$ is a positive way to say "$a \neq 0$".

A **positive** alternative to Booleanization

Given $(S, \lhd, Pos)$, the formula

$$\boxed{\forall z[Pos(x \wedge z) \Leftrightarrow Pos(y \wedge z)]}$$

defines a congruence, hence a sublocale, with the following properties:

- it is the smallest *strongly* dense sublocale (as defined by Johnstone);
- it is overt;
- it can be discrete (e. g. when the given topology is discrete).

These are precisely Sambin's **overlap algebras**.

A similar construction applies to σ-locales...

σ-sublocales

A **congruence** $\sim$ on a σ-frame L is
an equivalence relation compatible with finite meets and countable joins.

The quotient σ-frame $\boxed{L/\sim}$ *is* a σ-sublocale of L.

$L/\sim$ is **dense** if $(\forall x \in L)(x \sim 0 \;\Rightarrow\; x = 0)$

We call a σ-locale **overt** if its corresponding (σ-coherent) formal topology is overt.

The smallest strongly-dense σ-sublocale

Proposition

Given an overt σ-locale L, the formula $\boxed{\forall z[Pos(x \wedge z) \Leftrightarrow Pos(y \wedge z)]}$ defines the smallest strongly-dense σ-sublocale of L.

CLASSICALLY: these are Madden's *d-reduced* σ-frames.
CONSTRUCTIVELY: they are σ versions of overlap algebras.

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Proposition

A σ-locale L is a σ-overlap-algebra if and only if its corresponding (σ-coherent) formal topology is an overlap algebra.

CLASSICAL reading: L is d-reduced (Madden) if and only if the free frame over L is a complete Boolean algebra.

References

- B. Banaschewski, *The frame envelope of a σ-frame*, Quaestiones Mathematicae (1993).
- F. C., *Overlap Algebras as Almost Discrete Locales*, submitted (available on arXiv).
- F. C. and G. Sambin, *The overlap algebra of regular opens*, J. Pure Appl. Algebra (2010).
- F. C. and M. E. Maietti and P. Toto, *Constructive version of Boolean algebra*, Logic Journal of the IGPL (2012).
- J. J. Madden, *k-frames*, J. Pure Appl. Algebra (1991).
- M. E. Maietti, *A minimalist two-level foundation for constructive mathematics*, APAL (2009).
- M. E. Maietti and G. Sambin, *Toward a minimalist foundation for constructive mathematics*, (2005).
- A. Simpson, *Measure, randomness and sublocales*, Ann. Pure Appl. Logic (2012).

References

- B. Banaschewski, *The frame envelope of a σ-frame*, Quaestiones Mathematicae (1993).
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Merci beaucoup!