$\sigma$-locales and Booleanization in Formal Topology

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EU, planet Earth, Solar system, Milky Way . . .
σ-frames and σ-locales
(see Alex Simpson’s talk)

A σ-frame is a poset with:

- countable joins (including the empty join)
- and finite meets (including the empty meet)

in which binary meets distribute over countable joins.

σLoc = category of σ-frames and the opposite of σ-frame homomorphisms
**σ-frames and σ-locales**

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A **σ-frame** is a poset with:
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\[
\sigma \text{Loc} = \text{category of } \sigma\text{-frames and the opposite of } \sigma\text{-frame homomorphisms}
\]

**Aim of this talk:**

- to prove some facts about **σ-frames**
- in a constructive and predicative framework, namely Formal Topology,
  (which can be formalized in the Minimalist Foundation + ACω).

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But, what is a countable set? (constructively)

Some classically equivalent definitions for a set $S$:

- $S$ is either (empty or) finite or countably infinite;
- $S$ is either empty or enumerable;
- Either $S = \emptyset$ or there exists $\mathbb{N} \rightarrow S$ (onto).
- ...
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**Definition**

$S$ is **countable** if there exists $\mathbb{N} \rightarrow 1 + S$ with $S$ contained in the image

(see literature on Synthetic Topology: Andrej Bauer, Davorin Lešnik).

$S$ is countable $\iff$ there exists $D \rightarrow S$ with $D \subseteq \mathbb{N}$ detachable

(see Bridges-Richman Varieties... 1987).
The set of countable subsets

Given a set $S$, we write $\mathcal{P}_{\omega_1}(S)$ for the set of countable subsets of $S$.

$$\mathcal{P}_{\omega_1}(S) \cong (1 + S)^\mathbb{N}/\sim$$

where $f \sim g$ means $S \cap f[\mathbb{N}] = S \cap g[\mathbb{N}]$. 

Some properties of $\mathcal{P}_{\omega_1}(S)$:

- $\mathcal{P}_{\omega_1}(S)$ is closed under countable joins ($\text{AC}_\omega$).
- If equality in $S$ is decidable, then $\mathcal{P}_{\omega_1}(S)$ is a $\sigma$-frame.

$\mathcal{P}_{\omega_1}(1)$ = "open" truth values (Rosolini's dominance) = free $\sigma$-frame on no generators = terminal $\sigma$-locale.
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  - = free $\sigma$-frame on no generators
  - = terminal $\sigma$-locale.
σ-locales in Formal Topology

Let $L$ be a σ-locale.

For $a \in L$ and $U \subseteq L$ define

\[
\prec_L U \overset{\text{def}}{\iff} a \leq \bigsqcup W \text{ for some countable } W \subseteq U.
\]

$\prec_L$ is a **cover relation** (Formal Topology), that is,

\[
\begin{align*}
  a \in U \quad &\quad a \prec U \quad \forall b \in U. b \prec V \\
  &\quad a \prec V \\
  a \prec U \quad &\quad a \land c \prec \{ b \land c \mid b \in U \} \\
  &\quad a \prec \{ \top \}
\end{align*}
\]
σ-locales in Formal Topology

Let \( L \) be a σ-locale.

For \( a \in L \) and \( U \subseteq L \) define

\[
a \triangleleft_L U \iff a \leq \bigvee W \text{ for some countable } W \subseteq U.
\]

\( \triangleleft_L \) is a cover relation (Formal Topology), that is,

\[
\begin{align*}
  a \in U & \quad \frac{}{a \triangleleft U} \\
  a \triangleleft U \quad \forall b \in U. b \triangleleft V & \quad \frac{a \triangleleft U}{a \triangleleft V} \\
  a \land c \triangleleft \{ b \land c \mid b \in U \} & \quad \frac{a \triangleleft U}{a \triangleleft \{ \top \}}
\end{align*}
\]

**Proposition**

\((L, \triangleleft_L, \land, \top)\) is (a predicative presentation of) the free frame over the σ-frame \( L \).

(cf. Banashewski, *The frame envelope of a σ-frame*, and Madden, *k-frames*)
Lindelöf elements in a frame

An element $a$ of a frame $F$ is **Lindelöf** if for every $U \subseteq F$

$$a \leq \bigvee U \implies a \leq \bigvee W$$

for some countable $W \subseteq U$.

Lindelöf elements are closed under countable joins (not under finite meets, in general).
Lindelöf elements in a frame

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**σ-coherent frame =**

- Lindelöf elements are closed under finite meets
  (and hence they form a σ-frame), and
- every element is a (non necessarily countable) join of Lindelöf elements.
\(\sigma\text{-coherent formal topologies}\)

\(\sigma\text{-coherent frames can be presented as formal topologies} (S, \triangleleft, \wedge, \top) \text{ where}\)

\[
a \triangleleft U \implies a \triangleleft \mathcal{W} \text{ for some countable } \mathcal{W} \subseteq U
\]
σ-coherent formal topologies

σ-coherent frames can be presented as formal topologies \((S, \triangleleft, \wedge, \top)\) where

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a \triangleleft U \implies a \triangleleft W \quad \text{for some countable } W \subseteq U
\]

Proposition

Given a σ-locale \(L\),

\((L, \triangleleft, \wedge, \top)\) is σ-coherent and

its σ-frame of Lindelöf elements is \(L\)

So σ-locales can be seen as σ-coherent formal topologies

(with a suitable notion of morphism).
Examples

Examples of $\sigma$-coherent formal topologies:

point-free versions of

- Cantor space $2^\mathbb{N}$
- Baire space $\mathbb{N}^\mathbb{N}$
- $S^\mathbb{N}$ with $S$ countable.

So their Lindelöf elements provide examples of $\sigma$-locales.
Dense sublocales

A congruence $\sim$ on a frame $L$ is an equivalence relation compatible with finite meets and arbitrary joins.

The quotient frame $L/\sim$ is a sublocale of $L$. 

Some well-known fact about dense sublocales: the "intersection" of dense sublocales is always dense (!), hence every locale contains a smallest dense sublocale which turns out to be a complete Boolean algebra ("Booleanization"); the corresponding congruence $x \sim y$ is $\forall z (y \land z = 0 \iff x \land z = 0)$. 

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$\sigma$-FormalTopology

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Dense sublocales

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$L/\sim$ is dense if $(\forall x \in L)(x \sim 0 \Rightarrow x = 0)$

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- the “intersection” of dense sublocales is always dense (!), hence
- every locale contains a smallest dense sublocale
- which turns out to be a complete Boolean algebra (“Booleanization”);
- the corresponding congruence $x \sim y$ is $\forall z(y \land z = 0 \iff x \land z = 0)$
Boolean locales are good but...

- non-trivial discrete locales are never Boolean
- Boolean locales have no points
- non-trivial Boolean locales are never overt

unless your logic is classical!

Recall that \((S, \triangleleft)\) is **overt** if there exists a predicate \(Pos\) such that

\[
\frac{Pos(a)}{\exists b \in U. Pos(b)} \quad \frac{\exists b \in U. Pos(b)}{a \triangleleft \{b \in U \mid Pos(b)\}}
\]

**INTUITION**: \(Pos(a)\) is a positive way to say \(a \neq 0\).
A positive alternative to Booleanization

Given \((S, \triangleleft, Pos)\), the formula

\[
\forall z [Pos(x \land z) \iff Pos(y \land z)]
\]

defines a congruence, hence a sublocale, with the following properties:

- it is the smallest *strongly* dense sublocale (as defined by Johnstone);
- it is overt;
- it can be discrete (e.g. when the given topology is discrete).

These are precisely Sambin’s overlap algebras.

A similar construction applies to \(\sigma\)-locales...
A congruence $\sim$ on a $\sigma$-frame $L$ is an equivalence relation compatible with finite meets and countable joins.

The quotient $\sigma$-frame $L/\sim$ is a $\sigma$-sublocale of $L$.

$L/\sim$ is dense if $(\forall x \in L)(x \sim 0 \Rightarrow x = 0)$

We call a $\sigma$-locale overt if its corresponding ($\sigma$-coherent) formal topology is overt.
The smallest strongly-dense $\sigma$-sublocale

**Proposition**

Given an overt $\sigma$-locale $L$, the formula $\forall z [\text{Pos}(x \land z) \iff \text{Pos}(y \land z)]$ defines the smallest strongly-dense $\sigma$-sublocale of $L$.

CLASSICALLY: these are Madden’s $d$-reduced $\sigma$-frames.
CONSTRUCTIVELY: they are $\sigma$ versions of overlap algebras.
The smallest strongly-dense $\sigma$-sublocale

**Proposition**

Given an overt $\sigma$-locale $L$, the formula $\forall z [\text{Pos}(x \land z) \iff \text{Pos}(y \land z)]$ defines the smallest strongly-dense $\sigma$-sublocale of $L$.

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**Proposition**

A $\sigma$-locale $L$ is a $\sigma$-overlap-algebra if and only if its corresponding ($\sigma$-coherent) formal topology is an overlap algebra.

CLASSICAL reading: $L$ is d-reduced (Madden) if and only if the free frame over $L$ is a complete Boolean algebra.


References


Merci beaucoup!